

# YAPA: A generic tool for computing intruder knowledge

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Reasoning about the knowledge of an attacker is a necessary step in many formal analyses of security protocols. In the framework of the applied pi calculus, as in similar languages based on equational logics, knowledge is typically expressed by two relations: deducibility and static equivalence. Several decision procedures have been proposed for these relations under a variety of equational theories. However, each theory has its particular algorithm, and none has been implemented so far.

We provide a generic procedure for deducibility and static equivalence that takes as input any convergent rewrite system. We show that our algorithm covers most of the existing decision procedures for convergent theories. We also provide an efficient implementation, and compare it briefly with the tools ProVerif and KiSs.

Categories and Subject Descriptors:  $\square$ :

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Additional Key Words and Phrases: Security protocols, deduction, static equivalence

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## 1. INTRODUCTION

Understanding security protocols often requires reasoning about the information accessible to an on-line attacker. Accordingly, many formal approaches to security rely on a notion of *deducibility* [Lowe 1996; Millen and Shmatikov 2001] that models whether a piece of data, typically a secret, is retrievable from a finite set of messages. Deducibility, however, does not always suffice to reflect the knowledge of an attacker. Consider for instance a protocol sending an encrypted Boolean value, say, a vote in an electronic voting protocol. Rather than deducibility, the key idea to express confidentiality of the plaintext is that an attacker should not be able to

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*distinguish* between the sequences of messages corresponding to each possible value. (Such security considerations typically motivate the use of randomized encryption.)

In the framework of the applied pi-calculus [Abadi and Fournet 2001], as in similar languages based on equational logics [Blanchet et al. 2008], indistinguishability corresponds to a relation called *static equivalence*: roughly, two sequences of messages are *statically equivalent* when they satisfy the same algebraic relations from the attacker’s point of view. Static equivalence plays an important role in the study of guessing attacks (e.g. [Corin et al. 2004; Baudet 2005; Abadi et al. 2006]), as well as for anonymity properties and electronic voting protocols (e.g. [Delaune et al. 2009]). Static equivalence is also used for specifying privacy in the context of RFID protocols [Arapinis et al. 2009]. In several cases, this notion has also been shown to imply the more complex and precise notion of cryptographic indistinguishability [Baudet et al. 2005; Abadi et al. 2006], related to probabilistic polynomial-time Turing machines. Two sequences of messages are *cryptographically indistinguishable* when their corresponding bit-string implementations are indistinguishable to any probabilistic polynomial-time Turing machine.

We emphasize that both deducibility and static equivalence apply to observations on finite sets of messages, and do not take into account the dynamic behavior of protocols. (This justifies the expression *static equivalence*.) Nevertheless, deducibility is used as a subroutine by many general decision procedures [Comon-Lundh and Shmatikov 2003; Chevalier et al. 2003]. Besides, it has been shown that observational equivalence in the applied pi-calculus coincides with labeled bisimulation [Abadi and Fournet 2001], that is, corresponds to checking a number of static equivalences and some standard bisimulation conditions.

Deducibility and static equivalence rely on an underlying equational theory for axiomatizing the properties of cryptographic functions. Many decision procedures [Abadi and Cortier 2006; Cortier and Delaune 2007] have been proposed to compute these relations under a variety of equational theories, including symmetric and asymmetric encryptions, signatures, exclusive OR, and homomorphic operators. However, except for the class of subterm convergent theories [Abadi and Cortier 2006], which covers the standard flavors of encryption and signature, each of these decision results introduces a new procedure, devoted to a particular theory. Even in the case of the general decidability criterion given in [Abadi and Cortier 2006], we note that the algorithm underlying the proof has to be adapted for each theory, depending on how the criterion is fulfilled.

Perhaps as a consequence of this fact, none of these decision procedures has been implemented so far. When we began this work, the only tool able to verify static equivalence was ProVerif [Blanchet 2001; Blanchet et al. 2008]. This general tool can handle various equational theories and analyze security protocols under active adversaries. However termination of the verifier is not guaranteed in general, and protocols are subject to (safe) approximations. Since then, a new tool, called KiSs, has been developed [Ciobăcă et al. 2009]. The procedure implemented in KiSs has many concepts in common with a preliminary version of this work [Baudet et al. 2009] but targets a different class of equational theories.

The present work aims to fill this gap between theory and implementation and

propose an efficient tool for deciding deducibility and static equivalence in a uniform way. It is initially inspired from a procedure for solving more general constraint systems related to active adversaries and equivalence of finite processes, presented in [Baudet 2005], with corrected extended version in [Baudet 2007] (in French). However, due to the complexity of the constraint systems, this decision procedure was only studied for subterm convergent theories, and remains too complex to enable an efficient implementation.

*Our Contributions.* In this paper, we provide and study a generic procedure for checking deducibility and static equivalence, taking as input any convergent theory (that is, any equational theory described by a finite convergent rewrite system). We prove the algorithm sound and complete, up to explicit failure cases. Note that (unfailing) termination cannot be guaranteed in general since the problem of checking deducibility and static equivalence is undecidable, even for convergent theories [Abadi and Cortier 2006]. To address this issue and turn our algorithm into a decision procedure for a given convergent theory, we provide two criteria. First, we define a syntactic criterion on the rewrite rules that ensures that the algorithm never fails. This criterion is enjoyed in particular by any convergent subterm theory, as well as the theories of blind signature and homomorphic encryption. Termination often follows from a simple analysis of the rules of the algorithm: as a proof of concept, we obtain a new decidability result for deducibility and static equivalence for the prefix theory, representing encryption in CBC mode. Second, we provide a termination criterion based on deducibility: provided that failure cannot occur, termination on a given input is equivalent to the existence of some natural finite representation of deducible terms. As a consequence, we obtain that our algorithm can decide deducibility and static equivalence for all the convergent theories shown to be decidable in [Abadi and Cortier 2006].

Our second contribution is an efficient implementation of this generic procedure, called YAPA. After describing the main features of the implementation, we report several experiments suggesting that our tool computes static equivalence faster and for more convergent theories than the general tool ProVerif [Blanchet 2001; Blanchet et al. 2008]. We also outline the main differences between YAPA and the recent tool Kiss.

*Outline.* We introduce our setting in Section 2, in particular the notion of term algebra and equational theory, that are used to model cryptographic primitives. Deducibility and static equivalence are defined in Section 3. We describe our procedure in Section 4 and prove its correctness and completeness in Section 5. We provide criteria for preventing failure in Section 6 and for ensuring termination in Section 7. The implementation of our procedure is discussed in Section 8. Some concluding remarks and perspectives can be found in Section 9. A number of technical proofs have been postponed to the appendix to ease the presentation.

## 2. PRELIMINARIES

### 2.1 Term algebra

We start by introducing the necessary notions to describe cryptographic messages in a symbolical way. For modeling cryptographic primitives, we assume given a

set of *function symbols*  $\mathcal{F}$  together with an arity function  $\text{ar} : \mathcal{F} \rightarrow \mathbb{N}$ . Symbols in  $\mathcal{F}$  of arity 0 are called *constants*. We consider a set of *variables*  $\mathcal{X}$  and a set of additional constants  $\mathcal{W}$  called *parameters*. The (usual, first-order) term algebra generated by  $\mathcal{F}$  over  $\mathcal{W}$  and  $\mathcal{X}$  is written  $\mathcal{F}[\mathcal{W} \cup \mathcal{X}]$  with elements denoted by  $T, U, T_1 \dots$ . More generally, we write  $\mathcal{F}'[A]$  for the least set of terms containing a set  $A$  and stable by application of symbols in  $\mathcal{F}' \subseteq \mathcal{F}$ .

We write  $\text{var}(T)$  (resp.  $\text{par}(T)$ ) for the set of variables (resp. parameters) that occur in a term  $T$ . These notations are extended to tuples and sets of terms in the usual way. The set of positions of a term  $T$  is written  $\text{pos}(T) \subseteq \mathbb{N}^*$ , and its set of subterms  $\text{st}(T)$ . The subterm of  $T$  at position  $p \in \text{pos}(T)$  is written  $T|_p$ . The term obtained by replacing  $T|_p$  with a term  $U$  in  $T$  is denoted  $T[U]_p$ .

A (*finite, partial*) *substitution*  $\sigma$  is a mapping from a finite subset of variables, called its *domain* and written  $\text{dom}(\sigma)$ , to terms. The *image* of a substitution is its image as a mapping  $\text{im}(\sigma) = \{\sigma(x) \mid x \in \text{dom}(\sigma)\}$ . Substitutions are extended to endomorphisms of  $\mathcal{F}[\mathcal{X} \cup \mathcal{W}]$  as usual. We use a postfix notation for their application. A term  $T$  (resp. a substitution  $\sigma$ ) is *ground* if  $\text{var}(T) = \emptyset$  (resp.  $\text{var}(\text{im}(\sigma)) = \emptyset$ ).

For our cryptographic purposes, it is useful to distinguish a subset  $\mathcal{F}_{\text{pub}}$  of  $\mathcal{F}$ , made of *public function symbols*, that is, intuitively, the symbols made available to the attacker. A *recipe* (or *second-order term*)  $M, N, M_1 \dots$  is a term in  $\mathcal{F}_{\text{pub}}[\mathcal{W} \cup \mathcal{X}]$ , that is, a term containing no *private* (non-public) function symbols. A *plain term* (or *first-order term*)  $t, r, s, t_1 \dots$  is a term in  $\mathcal{F}[\mathcal{X}]$ , that is, containing no parameters. A (*public, ground, non-necessarily linear*) *n-ary context*  $C$  is a recipe in  $\mathcal{F}_{\text{pub}}[\mathbf{w}_1, \dots, \mathbf{w}_n]$ , where we assume a fixed countable subset of parameters  $\{\mathbf{w}_1, \dots, \mathbf{w}_n, \dots\} \subseteq \mathcal{W}$ . If  $C$  is a *n-ary context*,  $C[T_1, \dots, T_n]$  denotes the term obtained by replacing each occurrence of  $\mathbf{w}_i$  with  $T_i$  in  $C$ .

## 2.2 Rewriting

A *rewrite system*  $\mathcal{R}$  is a finite set of *rewrite rules*  $l \rightarrow r$  where  $l, r \in \mathcal{F}[\mathcal{X}]$  and such that  $\text{var}(r) \subseteq \text{var}(l)$ . A term  $S$  *rewrites* to  $T$  by  $\mathcal{R}$ , denoted  $S \rightarrow_{\mathcal{R}} T$ , if there exist  $l \rightarrow r$  in  $\mathcal{R}$ ,  $p \in \text{pos}(S)$  and a substitution  $\sigma$  such that  $S|_p = l\sigma$  and  $T = S[r\sigma]_p$ . We write  $\rightarrow_{\mathcal{R}}^+$  for the transitive closure of  $\rightarrow_{\mathcal{R}}$ ,  $\rightarrow_{\mathcal{R}}^*$  for its reflexive and transitive closure, and  $=_{\mathcal{R}}$  for its reflexive, symmetric and transitive closure.

A rewrite system  $\mathcal{R}$  is *convergent* if it is:

- *terminating*, i.e. there is no infinite chains  $T_1 \rightarrow_{\mathcal{R}} T_2 \rightarrow_{\mathcal{R}} \dots$ ; and
- *confluent*, i.e. for every terms  $S, T$  such that  $S =_{\mathcal{R}} T$ , there exists  $U$  such that  $S \rightarrow_{\mathcal{R}}^* U$  and  $T \rightarrow_{\mathcal{R}}^* U$ .

A term  $T$  is  $\mathcal{R}$ -*reduced* if there is no term  $S$  such that  $T \rightarrow_{\mathcal{R}} S$ . If  $T \rightarrow_{\mathcal{R}}^* S$  and  $S$  is  $\mathcal{R}$ -reduced then  $S$  is a  $\mathcal{R}$ -*reduced form* of  $T$ . When this reduced form is unique (in particular if  $\mathcal{R}$  is convergent), we write  $S = T \downarrow_{\mathcal{R}}$  (or simply  $T \downarrow$  when  $\mathcal{R}$  is clear from the context).

## 2.3 Equational theories

We equip the signature  $\mathcal{F}$  with an equational theory represented by a set of equations  $\mathcal{E}$  of the form  $s = t$  with  $s, t \in \mathcal{F}[\mathcal{X}]$ . The equational theory  $\mathbf{E}$  generated by  $\mathcal{E}$

is the least set of equations containing  $\mathcal{E}$  that is stable under the axioms of congruence (reflexivity, symmetry, transitivity, application of function symbols) and under application of substitutions. We write  $=_{\mathbf{E}}$  for the corresponding relation on terms. Equational theories have proved very useful for modeling algebraic properties of cryptographic primitives (see e.g. [Cortier et al. 2006] for a survey).

We are particularly interested in theories  $\mathbf{E}$  that can be represented by a convergent rewrite system  $\mathcal{R}$ , i.e. theories for which there exists a convergent rewrite system  $\mathcal{R}$  such that the two relations  $=_{\mathcal{R}}$  and  $=_{\mathbf{E}}$  coincide. The rewrite system  $\mathcal{R}$ —and by extension the equational theory  $\mathbf{E}$ —is *weakly subterm convergent* if, in addition, we have that for every rule  $l \rightarrow r \in \mathcal{R}$ ,  $r$  is either a subterm of  $l$  or a ground  $\mathcal{R}$ -reduced term. This class encompasses the class of subterm convergent theories used in [Abadi and Cortier 2006] (for every rule  $l \rightarrow r \in \mathcal{R}$ ,  $r$  is a subterm of  $l$  or a constant), the class of dwindling theories used in [Anantharaman et al. 2007], and the class of public-collapsing theories introduced in [Delaune and Jacquemard 2004].

*Example 2.1.* Consider the signature  $\mathcal{F}_{\text{enc}} = \{\text{dec}, \text{enc}, \langle -, - \rangle, \text{proj}_1, \text{proj}_2\}$ . The symbols  $\text{dec}, \text{enc}$  and  $\langle -, - \rangle$  are functional symbols of arity 2 that represent respectively the decryption, encryption and pairing functions, whereas  $\text{proj}_1$  and  $\text{proj}_2$  are functional symbols of arity 1 that represent the projection function on the first and the second component of a pair, respectively. The equational theory of pairing and symmetric (deterministic) encryption, denoted by  $\mathbf{E}_{\text{enc}}$ , is generated by the equations

$$\mathcal{E}_{\text{enc}} = \{\text{dec}(\text{enc}(x, y), y) = x, \text{proj}_1(\langle x, y \rangle) = x, \text{proj}_2(\langle x, y \rangle) = y\}.$$

Motivated by the modeling of the ECB mode of encryption, we may also consider an encryption symbol that is homomorphic with respect to pairing:

$$\mathcal{E}_{\text{hom}} = \mathcal{E}_{\text{enc}} \cup \left\{ \begin{array}{l} \text{enc}(\langle x, y \rangle, z) = \langle \text{enc}(x, z), \text{enc}(y, z) \rangle \\ \text{dec}(\langle x, y \rangle, z) = \langle \text{dec}(x, z), \text{dec}(y, z) \rangle \end{array} \right\}.$$

If we orient the equations from left to right, we obtain two rewrite systems  $\mathcal{R}_{\text{enc}}$  and  $\mathcal{R}_{\text{hom}}$ . Both rewrite systems are convergent, only  $\mathcal{R}_{\text{enc}}$  is (weakly) subterm convergent. Other examples of subterm convergent theories can be found in [Abadi and Cortier 2006].

From now on, we assume given a equational theory  $\mathbf{E}$  represented by a convergent rewrite system  $\mathcal{R}$ . A symbol  $f$  is *free* if  $f$  does not occur in  $\mathcal{R}$ . In order to model (an unbounded number of) random values possibly generated by the attacker, we assume that  $\mathcal{F}_{\text{pub}}$  contains infinitely many free public constants. We will use free private constants to model secrets, for instance the secret keys used to encrypt a message. Private (resp. public) free constants are closely related to bound (resp. free) *names* in the framework of the applied pi calculus [Abadi and Fournet 2001]. Our formalism also allows one to consider non-constant private symbols.

### 3. DEDUCIBILITY AND STATIC EQUIVALENCE

In order to describe the cryptographic messages observed or inferred by an attacker, we introduce the following notions of deduction facts and frames.

A *deduction fact* is a pair, written  $M \triangleright t$ , made of a recipe  $M \in \mathcal{F}_{\text{pub}}[\mathcal{W} \cup \mathcal{X}]$  and a plain term  $t \in \mathcal{F}[\mathcal{X}]$ . Such a deduction fact is *ground* if  $\text{var}(M, t) = \emptyset$ . A *frame*, denoted by letters  $\varphi, \Phi, \Phi_0, \dots$ , is a finite set of ground deduction facts. The *image* of a frame is defined by  $\text{im}(\Phi) = \{t \mid M \triangleright t \in \Phi\}$ . A frame  $\Phi$  is *one-to-one* if  $M_1 \triangleright t, M_2 \triangleright t \in \Phi$  implies  $M_1 = M_2$ .

A frame  $\varphi$  is *initial* if it is of the form  $\varphi = \{w_1 \triangleright t_1, \dots, w_\ell \triangleright t_\ell\}$  for some distinct parameters  $w_1, \dots, w_\ell \in \mathcal{W}$ . The parameters  $w_i$  can be seen as labels that refer to the messages observed by an attacker. Initial frames are closely related to the notion of frames in the applied pi-calculus [Abadi and Fournet 2001]. The only difference is that, in initial frames, values initially unknown to an attacker are modeled by private constants while they are modeled by *restricted names* in the applied pi-calculus. Name generation and binding are important features of the (general) applied calculus but are unessential when considering finite processes, and in particular frames. Given such an initial frame  $\varphi$ , we denote by  $\text{dom}(\varphi)$  its *domain*  $\text{dom}(\varphi) = \{w_1, \dots, w_\ell\}$ . If  $\text{par}(M) \subseteq \text{dom}(\varphi)$ , we write  $M\varphi$  for the term obtained by replacing each  $w_i$  by  $t_i$  in  $M$ . We note that if in addition  $M$  is ground then  $t = M\varphi$  is a ground plain term.

### 3.1 Deducibility, recipes

Classically (see e.g. [Abadi and Cortier 2006]), a ground term  $t$  is *deducible* modulo  $\mathbf{E}$  from an initial frame  $\varphi$ , written  $\varphi \vdash_{\mathbf{E}} t$ , if there exists  $M \in \mathcal{F}_{\text{pub}}[\text{dom}(\varphi)]$  such that  $M\varphi =_{\mathbf{E}} t$ . This corresponds to the intuition that the attacker may compute (infer)  $t$  from  $\varphi$ . For the purpose of our study, we generalize this notion to arbitrary (i.e. non-necessarily initial) frames, and even sets of (non-necessarily ground) deduction facts  $\phi$ , using the notations  $\triangleright_\phi$  and  $\triangleright_\phi^{\mathbf{E}}$  defined as follows.

*Definition 3.1 Deducibility.* Let  $\phi$  be finite set of deductions facts. We say that  $M$  is a *recipe of  $t$  in  $\phi$* , written  $M \triangleright_\phi t$ , if there exist a (public, ground, non-necessarily linear)  $n$ -ary context  $C$  and some deduction facts  $M_1 \triangleright t_1, \dots, M_n \triangleright t_n$  in  $\phi$  such that  $M = C[M_1, \dots, M_n]$  and  $t = C[t_1, \dots, t_n]$ . In that case, we say that  $t$  is *syntactically deducible* from  $\phi$ , also written  $\phi \vdash t$ .

We say that  $M$  is a *recipe of  $t$  in  $\phi$  modulo  $\mathbf{E}$* , written  $M \triangleright_\phi^{\mathbf{E}} t$ , if there exists a term  $t'$  such that  $M \triangleright_\phi t'$  and  $t' =_{\mathbf{E}} t$ . In that case, we say that  $t$  is *deducible from  $\phi$  modulo  $\mathbf{E}$* , written  $\phi \vdash_{\mathbf{E}} t$ .

We note that  $M \triangleright_\varphi t$  is equivalent to  $M\varphi = t$  when  $\varphi$  is an initial frame and when  $t$  (or equivalently  $M$ ) is ground. We also note that in the case of a frame  $\varphi$ , since our contexts  $C$  are ground and public,  $M \triangleright_\varphi t$  implies  $\text{var}(M, t) = \emptyset$  and  $\text{par}(M) \subseteq \text{par}(\varphi)$ .

*Example 3.2.* Consider the equational theory  $\mathbf{E}_{\text{enc}}$  described in Example 2.1. Let  $\varphi_0 = \{w_1 \triangleright \text{enc}(c_0, k), w_2 \triangleright k\}$  where  $c_0$  is a public constant and  $k$  is a private constant. We have that  $\varphi_0$  is a set of deduction facts. Since, these facts are ground,  $\varphi_0$  is actually a frame. Moreover, this frame is initial. We have that  $\langle w_2, w_2 \rangle \triangleright_{\varphi_0} \langle k, k \rangle$ ,  $c_0 \triangleright_{\varphi_0} c_0$ , and  $\text{dec}(w_1, w_2) \triangleright_{\varphi_0}^{\mathbf{E}_{\text{enc}}} c_0$ .

### 3.2 Static equivalence, visible equations

Deducibility does not always suffice for expressing the knowledge of an attacker. In particular, it does not account for the partial information that an attacker may obtain about secrets. Sometimes, the attacker can deduce exactly the same set of terms from two different frames but he could still be able to tell the difference between these two frames. This issue motivates the study of visible equations and static equivalence (see [Abadi and Fournet 2001]), defined as follows.

*Definition 3.3 Static equivalence.* Let  $\varphi$  be an initial frame. The set of *visible equations of  $\varphi$  modulo  $E$*  is defined as

$$\text{eq}_E(\varphi) = \{M \bowtie N \mid M, N \in \mathcal{F}_{\text{pub}}[\text{dom}(\varphi)], M\varphi =_E N\varphi\}$$

where  $\bowtie$  is a dedicated commutative symbol. Two initial frames  $\varphi_1$  and  $\varphi_2$  with the same domain are *statically equivalent* modulo  $E$ , written  $\varphi_1 \approx_E \varphi_2$ , if their sets of visible equations are equal, i.e.  $\text{eq}_E(\varphi_1) = \text{eq}_E(\varphi_2)$ .

This definition is in line with static equivalence in the applied pi calculus [Abadi and Fournet 2001] where bounds names would be replaced by free private constants.

*Example 3.4.* Consider again the equational theory  $E_{\text{enc}}$  given in Example 2.1. Let  $\varphi_0 = \{w_1 \triangleright \text{enc}(c_0, k), w_2 \triangleright k\}$  and  $\varphi_1 = \{w_1 \triangleright \text{enc}(c_1, k), w_2 \triangleright k\}$  where  $c_0, c_1$  are public constants and  $k$  is a private constant. We have that:

- $(\text{enc}(c_0, w_2) \bowtie w_1) \in \text{eq}_{E_{\text{enc}}}(\varphi_0)$ , and
- $(\text{enc}(c_0, w_2) \bowtie w_1) \notin \text{eq}_{E_{\text{enc}}}(\varphi_1)$ .

Hence,  $\text{eq}_{E_{\text{enc}}}(\varphi_0) \neq \text{eq}_{E_{\text{enc}}}(\varphi_1)$  and the two frames  $\varphi_0$  and  $\varphi_1$  are not statically equivalent. However, it can be shown that  $\{w_1 \triangleright \text{enc}(c_0, k)\} \approx_{E_{\text{enc}}} \{w_1 \triangleright \text{enc}(c_1, k)\}$ .

For the purpose of finitely describing the set of visible equations  $\text{eq}_E(\varphi)$  of an initial frame, we introduce *quantified equations* of the form  $\forall z_1, \dots, z_q. M \bowtie N$  where  $z_1, \dots, z_q \in \mathcal{X}$ ,  $q \geq 0$  and  $\text{var}(M, N) \subseteq \{z_1, \dots, z_q\}$ . In what follows, finite sets of quantified equations are denoted  $\Psi, \Psi_0, \dots$ . We write  $\Psi \models M \bowtie N$  when the ground equation  $M \bowtie N$  is a consequence of  $\Psi$  in the usual, first-order logics with equality axioms for the relation  $\bowtie$  (that is, reflexivity, symmetry, transitivity and compatibility with symbols in  $\mathcal{F}_{\text{pub}}$ ). When no confusion arises, we may refer to quantified equations simply as *equations*. As usual, quantified equations are considered up to renaming of bound variables.

*Example 3.5.* Consider the equational theory  $E_{\text{hom}}$  given in Example 2.1. Let  $\varphi = \{w_1 \triangleright \text{enc}(\langle c_0, c_1 \rangle, k), w_2 \triangleright \langle \text{enc}(c_0, k), \text{enc}(c_1, k) \rangle, w_3 \triangleright k\}$  where  $c_0$  and  $c_1$  are public constants and  $k$  is a private constant. In the set  $\text{eq}_{E_{\text{hom}}}(\varphi)$ , we have, among others,  $w_1 \bowtie w_2$  and  $\text{dec}(w_1, M) \bowtie \langle \text{dec}(\text{proj}_1(w_1), M), \text{dec}(\text{proj}_2(w_1), M) \rangle$  for every term  $M \in \mathcal{F}_{\text{pub}}[\text{dom}(\varphi)]$ . Indeed, we have that:

$$\begin{aligned} \text{dec}(w_1, M)\varphi &= \text{dec}(\text{enc}(\langle c_0, c_1 \rangle, k), M\varphi) \\ &=_{E_{\text{hom}}} \langle \text{dec}(\text{enc}(c_0, k), M\varphi), \text{dec}(\text{enc}(c_1, k), M\varphi) \rangle \\ &=_{E_{\text{hom}}} \langle \text{dec}(\text{proj}_1(w_1), M), \text{dec}(\text{proj}_2(w_1), M) \rangle \varphi \end{aligned}$$

This infinite set will be represented with the quantified equation:

$$\forall z. \text{dec}(w_1, z) \bowtie \langle \text{dec}(\text{proj}_1(w_1), z), \text{dec}(\text{proj}_2(w_1), z) \rangle.$$



#### 4. MAIN PROCEDURE

In this section, we describe our algorithms for checking deducibility and static equivalence on convergent rewrite systems. After some additional notations, we present the core of the procedure, which consists of a set of transformation rules used to saturate a frame and a finite set of quantified equations. The result of the saturation can be seen as a finite description of the deducible terms and visible equations of the initial frame under consideration. We then show how to use this procedure to decide deducibility and static equivalence, provided that saturation succeeds. (Recall that static equivalence and deduction are undecidable for convergent theories [Abadi and Cortier 2006].)

Soundness and completeness of the saturation procedure are detailed in Section 5. We provide sufficient conditions on the rewrite systems to ensure success of saturation and termination in Section 6 and Section 7.

##### 4.1 Decompositions of rewrite rules

Before stating the procedure, we introduce the following notion of *decomposition* to account for the possible superpositions of an attacker's context (that is, a recipe in our setting) with a left-hand side of rewrite rule.

*Definition 4.1 Decomposition.* Let  $n, p, q$  be non-negative integers. A  $(n, p, q)$ -*decomposition* of a term  $l$  (and by an extension of any rewrite rule  $l \rightarrow r$ ) is a (public, ground, non-necessarily linear) context  $D \in \mathcal{F}_{\text{pub}}[\mathcal{W}]$  such that  $\text{par}(D) = \{\mathbf{w}_1, \dots, \mathbf{w}_{n+p+q}\}$  and  $l = D[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$  where

- $l_1, \dots, l_n$  are mutually-distinct non-variable terms,
- $y_1, \dots, y_p$  and  $z_1, \dots, z_q$  are mutually-distinct variables, and
- $y_1, \dots, y_p \in \text{var}(l_1, \dots, l_n)$  whereas  $z_1, \dots, z_q \notin \text{var}(l_1, \dots, l_n)$ .

A decomposition  $D$  is *proper* if it is not a parameter (i.e.  $D \neq \mathbf{w}_1$ ).

In order to avoid unnecessary computations,  $(n, p, q)$ -decompositions are considered up to permutations of parameters in the sets  $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ ,  $\{\mathbf{w}_{n+1}, \dots, \mathbf{w}_{n+p}\}$  and  $\{\mathbf{w}_{n+p+1}, \dots, \mathbf{w}_{n+p+q}\}$  respectively.

*Example 4.2.* Consider the rewrite rule  $\text{dec}(\text{enc}(x, y), y) \rightarrow x$ . This rule admits two proper decompositions up to permutation of parameters:

- $D_1 = \text{dec}(\text{enc}(\mathbf{w}_1, \mathbf{w}_2), \mathbf{w}_2)$  where  $n = 0, p = 0, q = 2, z_1 = x, z_2 = y$ ;
- $D_2 = \text{dec}(\mathbf{w}_1, \mathbf{w}_2)$  where  $n = 1, p = 1, q = 0, l_1 = \text{enc}(x, y)$  and  $y_1 = y$ .

Now, consider the rewrite rule  $\text{dec}(\langle x, y \rangle, z) \rightarrow \langle \text{dec}(x, z), \text{dec}(y, z) \rangle$ . This rule also admits two proper decompositions:

- $D_3 = \text{dec}(\langle \mathbf{w}_1, \mathbf{w}_2 \rangle, \mathbf{w}_3)$  where  $n = 0, p = 0, q = 3, z_1 = x, z_2 = y, z_3 = z$ ;
- $D_4 = \text{dec}(\mathbf{w}_1, \mathbf{w}_2)$  where  $n = 1, p = 0, q = 1, l_1 = \langle x, y \rangle, z_1 = z$ .

##### 4.2 Transformation rules

To check deducibility and static equivalence, we proceed by saturating an initial frame, adding some deduction facts and equations satisfied by the frame. We



**A. Inferring deduction facts and equations by context reduction**

Assume that

$$\begin{aligned} l &= D[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q] \text{ is a proper decomposition of } (l \rightarrow r) \in \mathcal{R} \\ M_1 \triangleright t_1, \dots, M_{n+p} \triangleright t_{n+p} &\in \Phi \\ (l_1, \dots, l_n, y_1, \dots, y_p) \sigma &= (t_1, \dots, t_{n+p}) \end{aligned}$$

- (1) If there exists  $M = \text{Ctx}(\Phi \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\} \vdash_{\mathcal{R}}^? r\sigma)$ , then

$$(\Phi, \Psi) \Longrightarrow (\Phi, \Psi \cup \{\forall z_1, \dots, z_q. D[M_1, \dots, M_{n+p}, z_1 \dots, z_q] \bowtie M\}) \quad (\mathbf{A.1})$$

- (2) Else, if  $(r\sigma)\downarrow_{\mathcal{R}}$  is ground, then

$$\begin{aligned} (\Phi, \Psi) \Longrightarrow & (\Phi \cup \{M_0 \triangleright (r\sigma)\downarrow_{\mathcal{R}}\}, \\ & \Psi \cup \{\forall z_1, \dots, z_q. D[M_1, \dots, M_{n+p}, z_1 \dots, z_q] \bowtie M_0\}) \end{aligned} \quad (\mathbf{A.2})$$

where  $M_0 = D[M_1, \dots, M_{n+p}, \mathbf{a}, \dots, \mathbf{a}]$  for some fixed public constant  $\mathbf{a}$ .

- (3) Otherwise,  $(\Phi, \Psi) \Longrightarrow \perp$  (\mathbf{A.3})

**B. Inferring deduction facts and equations syntactically**

Assume that  $M_0 \triangleright t_0, \dots, M_n \triangleright t_n \in \Phi \quad t = f(t_1, \dots, t_n) \in \text{st}(t_0) \quad f \in \mathcal{F}_{\text{pub}}$

- (1) If there exists  $M$  such that  $(M \triangleright t) \in \Phi$ ,

$$(\Phi, \Psi) \Longrightarrow (\Phi, \Psi \cup \{f(M_1, \dots, M_n) \bowtie M\}) \quad (\mathbf{B.1})$$

- (2) Otherwise,  $(\Phi, \Psi) \Longrightarrow (\Phi \cup \{f(M_1, \dots, M_n) \triangleright t\}, \Psi)$  (\mathbf{B.2})

Fig. 1. Transformation rules

consider *states* that are either the failure state  $\perp$  or a couple  $(\Phi, \Psi)$  formed by a one-to-one frame  $\Phi$  in  $\mathcal{R}$ -reduced form and a finite set of quantified equations  $\Psi$ .

Given an initial frame  $\varphi$ , our procedure starts from an initial state associated to  $\varphi$ , denoted by  $\text{Init}(\varphi)$ , obtained by reducing  $\varphi$  and replacing duplicated terms by equations. Formally,  $\text{Init}(\varphi)$  is the result of a procedure recursively defined as follows:  $\text{Init}(\emptyset) = (\emptyset, \emptyset)$ , and assuming  $\text{Init}(\varphi) = (\Phi, \Psi)$ , we have

$$\text{Init}(\varphi \uplus \{w \triangleright t\}) = \begin{cases} (\Phi, \Psi \cup \{w \bowtie w'\}) & \text{if there exists some } w' \triangleright t\downarrow_{\mathcal{R}} \in \Phi \\ (\Phi \cup \{w \triangleright t\downarrow_{\mathcal{R}}\}, \Psi) & \text{otherwise.} \end{cases}$$

*Example 4.3.* Consider the frames  $\varphi_0$ ,  $\varphi_1$  and  $\varphi$  introduced respectively in Example 3.4 and Example 3.5. We have that  $\text{Init}(\varphi_0) = (\varphi_0, \emptyset)$ ,  $\text{Init}(\varphi_1) = (\varphi_1, \emptyset)$  and  $\text{Init}(\varphi) = (\{w_1 \triangleright \langle \text{enc}(c_0, k), \text{enc}(c_1, k) \rangle, w_3 \triangleright k\}, \{w_1 \bowtie w_2\})$ .

The main part of our procedure consists in saturating a state  $(\Phi, \Psi)$  by means of the transformation rules described in Figure 1. The **A** rules are designed for applying a rewrite step on top of existing deduction facts. If the resulting term  $(r\sigma)\downarrow_{\mathcal{R}}$  is already deducible (in some specific sense that we make precise below) then a corresponding equation is added (rule **A.1**); or else if it is ground, the corresponding deduction fact is added to the state (rule **A.2**); otherwise, the procedure may fail (rule **A.3**). The **B** rules are meant to add syntactically deducible subterms (rule **B.2**) or related equations (rule **B.1**).

For technical reasons, rule **A.1** is parametrized by a function  $\text{Ctx}$  that outputs either a recipe  $M$  or the special symbol  $\perp$ . This function has to satisfy the following properties:

- (a) if  $\phi \vdash t \downarrow_{\mathcal{R}}$ , then  $\text{Ctx}(\phi \vdash_{\mathcal{R}}^? t) \neq \perp$ ;
- (b) if  $M = \text{Ctx}(\phi \vdash_{\mathcal{R}}^? t)$  then there exists  $s$  such that  $M \triangleright_{\phi} s$  and  $t \rightarrow_{\mathcal{R}}^* s$ . (This justifies the notation  $\phi \vdash_{\mathcal{R}}^? t$  used to denote a specific deducibility problem.)

Property (a) ensures that the rules transform a state into a state (and more precisely that the resulting frame in **(A.2)** is still one-to-one). Property (b) guarantees the soundness of the new equation in **(A.1)**. Requiring  $t \rightarrow_{\mathcal{R}}^* s$  instead  $t =_{\mathcal{E}} s$  is necessary for the proof of completeness. In what follows, a *function*  $\text{Ctx}$  is any function satisfying the two properties (a) and (b).

A simple choice for  $\text{Ctx}(\phi \vdash_{\mathcal{R}}^? t)$  is to solve the deducibility problem  $\phi \vdash_{\mathcal{R}}^? t \downarrow_{\mathcal{R}}$  in the empty equational theory, and then return a corresponding recipe  $M$ , if any. (This problem is easily solved by induction on  $t \downarrow_{\mathcal{R}}$ .) We will see in Section 6 that this choice is sufficient to avoid failure for a large class of equational theories, namely the class of layered convergent theories. However the proof of this fact relies on an intermediate result that uses a different choice of  $\text{Ctx}$ .

*Example 4.4.* Consider the frame  $\varphi_0$  previously described in Example 3.4. We can apply rule **A.1** as follows. Consider the rewrite rule  $\text{dec}(\text{enc}(x, y), y) \rightarrow x$ , the decomposition  $D_2$  given in Example 4.2 and  $t_1 = \text{enc}(c_0, k)$ . We have that  $\text{Init}(\varphi_0) = (\varphi_0, \emptyset) \Rightarrow (\varphi_0, \{\text{dec}(w_1, w_2) \bowtie c_0\})$ . In other words, since we know the key  $k$  through  $w_2$ , we can check that the decryption of  $w_1$  by  $w_2$  leads to the public constant  $c_0$ . Next we apply rule **B.1** as follows:

$$(\varphi_0, \{\text{dec}(w_1, w_2) \bowtie c_0\}) \Rightarrow (\varphi_0, \{\text{dec}(w_1, w_2) \bowtie c_0, \text{enc}(c_0, w_2) \bowtie w_1\}).$$

No more rules can then modify the state. Similarly for  $\varphi_1$ , we obtain that:

$$\begin{aligned} \text{Init}(\varphi_1) &= (\varphi_1, \emptyset) \\ &\Rightarrow (\varphi_1, \{\text{dec}(w_1, w_2) \bowtie c_1\}) \\ &\Rightarrow (\varphi_1, \{\text{dec}(w_1, w_2) \bowtie c_1, \text{enc}(c_1, w_2) \bowtie w_1\}). \end{aligned}$$

*Example 4.5.* Consider the frame  $\varphi$  described in Example 3.5. We can apply rule **A.1** as follows. Consider the rewrite rule  $\text{dec}(\langle x, y \rangle, z) \rightarrow \langle \text{dec}(x, z), \text{dec}(y, z) \rangle$ , the decomposition  $D_4$  given in Example 4.2 and  $t_1 = \langle \text{enc}(c_0, k), \text{enc}(c_1, k) \rangle$ . We have that  $r\sigma = \langle \text{dec}(\text{enc}(c_0, k), z_1), \text{dec}(\text{enc}(c_1, k), z_1) \rangle$ , and thus  $\text{Init}(\varphi) \Rightarrow \perp$ . We have that  $r\sigma \downarrow_{\mathcal{R}} = r\sigma$ . The condition required in case (1) is not fulfilled and the condition stated in case (2) is false.

However, note that another strategy of rules application allows us to consider this decomposition. For this, it is sufficient to apply first **B** rules to add the deduction facts  $\text{proj}_1(w_1) \triangleright \text{enc}(c_0, k)$  and  $\text{proj}_2(w_1) \triangleright \text{enc}(c_1, k)$ . Now, we have that  $r\sigma \downarrow_{\mathcal{R}}$  is syntactically deducible: the condition required in case (1) is full-filled and we finally add the equation:  $\forall z_1. \text{dec}(w_1, z_1) \bowtie \langle \text{dec}(\text{proj}_1(w_1), z_1), \text{dec}(\text{proj}_2(w_1), z_1) \rangle$ .

We write  $\Rightarrow^*$  for the transitive and reflexive closure of  $\Rightarrow$ . The definitions of  $\text{Ctx}$  and of the transformation rules ensure that whenever  $S \Rightarrow^* S'$  and  $S$  is a state, then  $S'$  is also a state, with the same parameters unless  $S' = \perp$ .

### 4.3 Main theorem

We now state the soundness and the completeness of the transformation rules provided that a *saturated state* is reached, that is, a state  $S \neq \perp$  such that  $S \Rightarrow S'$

implies  $S' = S$ . The technical lemmas involved in the proof of this theorem are detailed in Section 5.

**THEOREM 4.6 SOUNDNESS AND COMPLETENESS.** *Let  $\mathbf{E}$  be an equational theory generated by a convergent rewrite system  $\mathcal{R}$ . Let  $\varphi$  be an initial frame and  $(\Phi, \Psi)$  be a saturated state such that  $\text{Init}(\varphi) \Longrightarrow^* (\Phi, \Psi)$ .*

(1) *For all  $M \in \mathcal{F}_{\text{pub}}[\text{par}(\varphi)]$  and  $t \in \mathcal{F}[\emptyset]$ , we have that:*

$$M\varphi =_{\mathbf{E}} t \iff \exists N \text{ such that } \Psi \models M \bowtie N \text{ and } N \triangleright_{\Phi} t \downarrow_{\mathcal{R}}.$$

(2) *For all  $M, N \in \mathcal{F}_{\text{pub}}[\text{par}(\varphi) \cup \mathcal{X}]$ , we have that:*

$$M\varphi =_{\mathbf{E}} N\varphi \iff \Psi \models M \bowtie N.$$

We note that this theorem applies to any saturated state reachable from the initial frame. Moreover, while the saturation procedure is sound and complete, it may not terminate, or it may *fail* if rule **A.3** becomes the only applicable rule at some point of computation. In Section 6 and Section 7, we explore several sufficient conditions to prevent failure and ensure termination.

#### 4.4 Application to deduction and static equivalence

Decision procedures for deduction and static equivalence modulo  $\mathbf{E}$  follow from Theorem 4.6.

*Algorithm for deduction.* Let  $\varphi$  be an initial frame and  $t$  be a ground term. The procedure for checking  $\varphi \vdash_{\mathbf{E}} t$  runs as follows:

- (1) Apply the transformation rules to obtain (if any) a saturated state  $(\Phi, \Psi)$  such that  $\text{Init}(\varphi) \Longrightarrow^* (\Phi, \Psi)$ ;
- (2) Return *yes* if there exists  $N$  such that  $N \triangleright_{\Phi} t \downarrow_{\mathcal{R}}$  (that is, the  $\mathcal{R}$ -reduced form of  $t$  is syntactically deducible from  $\Phi$ ); otherwise return *no*.

**PROOF.** If the algorithm returns *yes*, this means that there exists  $N$  such that  $N \triangleright_{\Phi} t \downarrow_{\mathcal{R}}$ . Thanks to Theorem 4.6 (1), we have that  $N\varphi =_{\mathbf{E}} t$ , i.e.  $N \triangleright_{\varphi}^{\mathbf{E}} t$ .

Conversely, if  $t$  is deducible from  $\varphi$ , then there exists  $M$  such that  $M\varphi =_{\mathbf{E}} t$ . By Theorem 4.6 (1), there exists  $N$  such that  $N \triangleright_{\Phi} t \downarrow_{\mathcal{R}}$ . The algorithm returns *yes*.  $\square$

*Example 4.7.* Consider the frame  $\varphi_0 = \{w_1 \triangleright \text{enc}(c_0, k), w_2 \triangleright k\}$  introduced in Example 3.2 and let  $t_1 = \langle k, k \rangle$  and  $t_2 = c_0$ . Let  $(\Phi_0, \Psi_0)$  be the saturated state described in Example 4.4. We have that:

$$(\Phi_0, \Psi_0) = (\varphi_0, \{\text{dec}(w_1, w_2) \bowtie c_0, \text{enc}(c_0, w_2) \bowtie w_1\}).$$

Then, it is easy to see that our algorithm for deduction will return *yes* for both terms  $t_1$  and  $t_2$ . Indeed, those terms are syntactically deducible from  $\varphi_0$ .

*Algorithm for static equivalence.* Let  $\varphi_1$  and  $\varphi_2$  be two initial frames. The procedure for checking  $\varphi_1 \approx_{\mathbf{E}} \varphi_2$  runs as follows:

- (1) Apply the transformation rules to obtain (if possible) two saturated states  $(\Phi_1, \Psi_1)$  and  $(\Phi_2, \Psi_2)$  such that  $\text{Init}(\varphi_i) \Longrightarrow^* (\Phi_i, \Psi_i)$ ,  $i = 1, 2$ ;

- (2) For  $\{i, j\} = \{1, 2\}$ , for every equation  $(\forall z_1, \dots, z_\ell. M \bowtie N)$  in  $\Psi_i$ , check that  $M\varphi_j =_{\mathbf{E}} N\varphi_j$  — that is, in other words,  $(M\varphi_j) \downarrow_{\mathcal{R}} = (N\varphi_j) \downarrow_{\mathcal{R}}$ ;
- (3) If so return *yes*; otherwise return *no*.

PROOF. If the algorithm returns *yes*, this means that  $M\varphi_2 =_{\mathbf{E}} N\varphi_2$  for every equation  $(\forall z_1, \dots, z_\ell. M \bowtie N)$  in  $\Psi_1$ . Let  $M \bowtie N \in \text{eq}_{\mathbf{E}}(\varphi_1)$ . By definition of  $\text{eq}_{\mathbf{E}}(\varphi_1)$ , we have that  $M\varphi_1 =_{\mathbf{E}} N\varphi_1$ . Thanks to Theorem 4.6 (2), we have that  $\Psi_1 \models M \bowtie N$ . As all the equations in  $\Psi_1$  are satisfied by  $\varphi_2$  modulo  $\mathbf{E}$ , we deduce that  $M\varphi_2 =_{\mathbf{E}} N\varphi_2$ , i.e.  $M \bowtie N \in \text{eq}(\varphi_2)$ . The other inclusion,  $\text{eq}_{\mathbf{E}}(\varphi_2) \subseteq \text{eq}_{\mathbf{E}}(\varphi_1)$ , is proved in the same way.

Conversely, assume now that  $\varphi_1 \approx_{\mathbf{E}} \varphi_2$ , i.e.  $\text{eq}_{\mathbf{E}}(\varphi_1) = \text{eq}_{\mathbf{E}}(\varphi_2)$ . Consider a quantified equation  $\forall z_1, \dots, z_\ell. M \bowtie N$  in  $\Psi_1$  and let us show that  $M\varphi_2 =_{\mathbf{E}} N\varphi_2$ . (The other case is done in a similar way, and we will conclude that the algorithm returns *yes*.) Let  $c_1, \dots, c_\ell$  be free public constants not occurring in  $M$  and  $N$ , and let  $(M', N') = (M, N)\{z_1 \mapsto c_1, \dots, z_\ell \mapsto c_\ell\}$ . Since  $\Psi_1 \models M' \bowtie N'$ , by Theorem 4.6 (2), we have that  $M'\varphi_1 =_{\mathbf{E}} N'\varphi_1$ . Besides,  $M'$  and  $N'$  are ground and  $\text{par}(M', N') \subseteq \text{par}(\Psi_1) \subseteq \text{par}(\varphi_1)$ . Thus,  $(M' \bowtie N') \in \text{eq}_{\mathbf{E}}(\varphi_1) \subseteq \text{eq}_{\mathbf{E}}(\varphi_2)$  and  $M'\varphi_2 =_{\mathbf{E}} N'\varphi_2$ . As the constants  $c_1, \dots, c_\ell$  are free in  $\mathbf{E}$  and do not occur in  $M$  and  $N$ , by replacement, we obtain that  $M\varphi_2 =_{\mathbf{E}} N\varphi_2$ .  $\square$

*Example 4.8.* Consider the frames  $\varphi_i = \{\mathbf{w}_1 \triangleright \text{enc}(c_i, k), \mathbf{w}_2 \triangleright k\}$  introduced in Example 3.4. Let  $(\Phi_0, \Psi_0)$  and  $(\Phi_1, \Psi_1)$  be the two saturated states described in Example 4.4. We have that  $\text{dec}(\mathbf{w}_1, \mathbf{w}_2) \bowtie c_0 \in \Psi_0$ , and

$$(\text{dec}(\mathbf{w}_1, \mathbf{w}_2)\varphi_1 =_{\mathbf{E}_{\text{enc}}} c_1 \neq_{\mathbf{E}_{\text{enc}}} c_0 = c_0\varphi_1.$$

Hence, our algorithm returns *no*. The two frames  $\varphi_0$  and  $\varphi_1$  are not statically equivalent.

## 5. SOUNDNESS AND COMPLETENESS OF THE SATURATION

The goal of this section is to prove Theorem 4.6. Section 5.1 is devoted to establish soundness of our saturation procedure, i.e. the  $\Leftarrow$  direction of Theorem 4.6. Showing the other direction, i.e. completeness, is more involved and is detailed in Section 5.2.

### 5.1 Soundness

First, the transformation rules are sound in the sense that, along the saturation process, we add only deducible terms and valid equations with respect to the initial frame.

LEMMA 5.1 SOUNDNESS. *Let  $\varphi$  be an initial frame and  $(\Phi, \Psi)$  be a state such that  $\text{Init}(\varphi) \Longrightarrow^* (\Phi, \Psi)$ . Then, we have that*

- (1)  $M \triangleright_{\Phi} t \Rightarrow M\varphi =_{\mathbf{E}} t$  for all  $M \in \mathcal{F}_{\text{pub}}[\text{dom}(\varphi)]$  and  $t \in \mathcal{F}[\emptyset]$ ;
- (2)  $\Psi \models M \bowtie N \Rightarrow M\varphi =_{\mathbf{E}} N\varphi$  for all  $M, N \in \mathcal{F}_{\text{pub}}[\text{dom}(\varphi) \cup \mathcal{X}]$ .

PROOF. We prove this result by induction on the derivation  $\text{Init}(\varphi) \Longrightarrow^* (\Phi, \Psi)$ .

*Base case:* We have that  $(\Phi, \Psi) = \text{Init}(\varphi)$  and we easily conclude.

*Induction case:* In such a case, we have  $\text{Init}(\varphi) \Longrightarrow^* (\Phi', \Psi') \Longrightarrow (\Phi, \Psi)$ .

Let us first notice two facts.

- (1) Let  $M$  and  $t$  be such that  $M \triangleright_{\Phi} t$ . By definition of  $\triangleright_{\Phi}$ , there exist a public context  $C$  and some deduction facts  $M'_1 \triangleright t'_1, \dots, M'_n \triangleright t'_n \in \Phi$  such that  $M = C[M'_1, \dots, M'_n]$  and  $t = C[t'_1, \dots, t'_n]$ . In order to prove 1., it is sufficient to show that  $M' \triangleright_{\Phi}^E t'$  for every  $M' \triangleright t' \in \Phi$ . By induction hypothesis, this holds for the deduction facts in  $\Phi'$ , thus it remains to show that  $M' \triangleright_{\Phi}^E t'$  for every fact  $M' \triangleright t' \in \Phi - \Phi'$ .
- (2) Let  $M, N$  be two terms such that  $\Psi \models M \bowtie N$ . To establish 2., it is sufficient to prove that  $M'\varphi =_E N'\varphi$  for every  $(\forall z_1, \dots, z_q. M' \bowtie N')$  in  $\Psi$ . By induction hypothesis, this holds for the equations in  $\Psi'$ , thus it remains to show that  $M'\varphi =_E N'\varphi$  for every equation  $(\forall z_1, \dots, z_q. M' \bowtie N')$  in  $\Psi - \Psi'$ .

Next we perform a case analysis on the inference rule used in  $(\Phi', \Psi') \implies (\Phi, \Psi)$ .

First, consider the case of rule **A**. Let  $l \rightarrow r \in \mathcal{R}$  be the rewrite rule,  $D$  the decomposition, and  $M_1 \triangleright t_1, \dots, M_{n+p} \triangleright t_{n+p}$  the facts involved in this step.

*Rule A.2:* We need to show that

$$\begin{aligned} & -D[M_1, \dots, M_{n+p}, \mathbf{a}, \dots, \mathbf{a}]\varphi =_E (r\sigma)\downarrow_{\mathcal{R}}, \text{ and} \\ & -D[M_1, \dots, M_{n+p}, z_1, \dots, z_q]\varphi =_E D[M_1, \dots, M_{n+p}, \mathbf{a}, \dots, \mathbf{a}]\varphi. \end{aligned}$$

We note that  $D[t_1, \dots, t_{n+p}, z_1, \dots, z_q] = l\sigma \rightarrow r\sigma \rightarrow^* (r\sigma)\downarrow_{\mathcal{R}}$ . Besides, by induction hypothesis we have that  $M_i\varphi =_E t_i$  for  $1 \leq i \leq n+p$ . Given that  $(r\sigma)\downarrow_{\mathcal{R}}$  is ground, and applying the substitution  $\{z_1 \mapsto \mathbf{a}, \dots, z_q \mapsto \mathbf{a}\}$  to the equation  $D[t_1, \dots, t_{n+p}, z_1, \dots, z_q] =_E (r\sigma)\downarrow_{\mathcal{R}}$ , we obtain:

$$\begin{aligned} D[M_1, \dots, M_{n+p}, z_1, \dots, z_q]\varphi & =_E D[t_1, \dots, t_{n+p}, z_1, \dots, z_q] \\ & =_E (r\sigma)\downarrow_{\mathcal{R}} \\ & =_E D[t_1, \dots, t_{n+p}, \mathbf{a}, \dots, \mathbf{a}] \\ & =_E D[M_1, \dots, M_{n+p}, \mathbf{a}, \dots, \mathbf{a}]\varphi \end{aligned}$$

*Rule A.1:* We need to show  $D[M_1, \dots, M_{n+p}, z_1, \dots, z_q]\varphi =_E M\varphi$ . As before, we have  $D[M_1, \dots, M_{n+p}, z_1, \dots, z_q]\varphi =_E (r\sigma)\downarrow_{\mathcal{R}}$ . We also know that there exists  $s$  such that  $M \triangleright_{\Phi^+} s$  and  $r\sigma \rightarrow_{\mathcal{R}}^* s$  where  $\Phi^+ = \Phi \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\}$  thanks to property (b) of Ctx. Let  $\theta$  be the substitution  $\{z_1 \mapsto \mathbf{a}, \dots, z_q \mapsto \mathbf{a}\}$ . We have that  $M\theta \triangleright_{\Phi} s$ . Hence, using the induction hypothesis, we have that  $M\theta\varphi =_E s$  thus  $M\varphi =_E s$ , i.e.  $M\varphi =_E (r\sigma)\downarrow_{\mathcal{R}}$ . This allows us to conclude.

*Rule A.3:* In such a case, the result trivially holds.

Second, we consider the case of **B** rules. Let  $t = f(t_1, \dots, t_n) \in \text{st}(t_0)$ ,  $f \in \mathcal{F}_{\text{pub}}$  and  $M_0 \triangleright t_0, \dots, M_n \triangleright t_n \in \Phi$  be involved in the step  $(\Phi', \Psi') \implies (\Phi, \Psi)$ .

*Rule B.1:* By induction hypothesis,  $M_i\varphi =_E t_i$  for every  $1 \leq i \leq n$ , hence  $f(M_1, \dots, M_n)\varphi =_E f(t_1, \dots, t_n) = t$ .

*Rule B.2:* By induction hypothesis,  $M_i\varphi =_E t_i$  for every  $1 \leq i \leq n$  and  $M\varphi =_E t$ , hence  $f(M_1, \dots, M_n)\varphi =_E f(t_1, \dots, t_n) = t =_E M\varphi$ .  $\square$

## 5.2 Completeness

The next three lemmas are dedicated to the completeness of **B** rules (Lemma 5.2 and Lemma 5.3) and **A** rules (Lemma 5.4).

Lemma 5.2 ensures that a saturated state  $(\Phi, \Psi)$  contains all the deduction facts  $M \triangleright t$  where  $t$  is a subterm of  $\Phi$  that is syntactically deducible, whereas Lemma 5.3 ensures that saturated states account for all the syntactic equations possibly visible on the frame.

LEMMA 5.2 COMPLETENESS, SYNTACTIC DEDUCTION. *Let  $(\Phi, \Psi)$  be a state,  $M_0 \triangleright t_0 \in \Phi$ . Let  $N, t$  be two terms such that  $t \in \text{st}(t_0)$  and  $N \triangleright_\Phi t$ . Then there exists  $(\Phi', \Psi')$  and  $N'$  such that:*

- $(\Phi, \Psi) \Longrightarrow^* (\Phi', \Psi')$  using **B** rules, and
- $N' \triangleright t \in \Phi'$  and  $\Psi' \models N \bowtie N'$ .

The proof of Lemma 5.2 is postponed to the appendix. It uses a simple induction on the context  $C$  witnessing the fact that  $t$  is syntactically deducible from  $\Phi$ .

LEMMA 5.3 COMPLETENESS, SYNTACTIC EQUATIONS. *Let  $(\Phi, \Psi)$  be a state, and  $M, N$  be two terms such that  $M \triangleright_\Phi t$  and  $N \triangleright_\Phi t$  for some term  $t$ . Then there exists  $(\Phi', \Psi')$  such that:*

- $(\Phi, \Psi) \Longrightarrow^* (\Phi', \Psi')$  using **B** rules, and
- $\Psi' \models M \bowtie N$ .

PROOF. (sketch) Let  $C, C'$  be the contexts witnessing  $M \triangleright_\Phi t$  and  $N \triangleright_\Phi t$ . Assume that  $C$  is smaller than  $C'$ . The proof is done by induction on  $C$ . When  $C$  is reduced to an hole, we apply Lemma 5.2 to conclude. Otherwise, we have that  $C = f(C_1, \dots, C_r)$  and  $C' = f(C'_1, \dots, C'_r)$ . We easily conclude by applying our induction hypothesis on  $C_i, C'_i$  for each  $1 \leq i \leq r$ . The detailed proof is presented in appendix A.  $\square$

Now, we know that terms that are syntactically deducible from the frame and syntactic equation visible on the frame will be added during our saturation procedure. It remains to take into account the underlying equational theory. This is the purpose of Lemma 5.4 that deals with the reduction of a deducible term along the rewrite system  $\mathcal{R}$ . Using that  $\mathcal{R}$  is convergent, this allows us to prove that every deducible term from a saturated frame is syntactically deducible.

LEMMA 5.4 COMPLETENESS, CONTEXT REDUCTION. *Let  $(\Phi, \Psi)$  be a state and  $M, t, t'$  be three terms such that  $M \triangleright_\Phi t$  and  $t \rightarrow_{\mathcal{R}} t'$ . Then, either  $(\Phi, \Psi) \Longrightarrow^* \perp$  or there exist  $(\Phi', \Psi'), M'$  and  $t''$  such that*

- $(\Phi, \Psi) \Longrightarrow^* (\Phi', \Psi')$ ,
- $M' \triangleright_{\Phi'} t''$  with  $t' \rightarrow_{\mathcal{R}}^* t''$ , and
- $\Psi' \models M \bowtie M'$ .

Besides, in both cases, the corresponding derivation from  $(\Phi, \Psi)$  can be chosen to consist of a number of **B** rules, possibly followed by one instance of **A** rule involving the same rewrite rule  $l \rightarrow r$  as the rewrite step  $t \rightarrow_{\mathcal{R}} t'$ .

PROOF. (sketch) The detailed proof of Lemma 5.4 is left to the appendix. We describe here its main arguments. Since  $t \rightarrow_{\mathcal{R}} t'$ , there exist a position  $\alpha$ , a substitution  $\sigma$  and a rewrite rule  $l \rightarrow r \in \mathcal{R}$  such that  $t|_{\alpha} = l\sigma$  and  $t' = t[r\sigma]_{\alpha}$ . Let  $C$  be

a context witnessing the fact that  $M \triangleright_{\Phi} t$ . Since terms in  $\text{im}(\Phi)$  are  $\mathcal{R}$ -reduced,  $\alpha$  is actually a position in  $C$ . Thus, the rewriting step mentioned above corresponds to a proper  $(n, p, q)$ -decomposition  $D$  of  $l$ :  $l = D[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$ . We can show that  $M|_{\alpha} \triangleright_{\Phi} l\sigma$  and  $D[M_1, \dots, M_n, N_1, \dots, N_{p+q}] \triangleright_{\Phi} l\sigma$  where

- $M_1 \triangleright t_1, \dots, M_n \triangleright t_n$  are deduction facts in  $\Phi$ ,
- for every  $1 \leq j \leq p$ ,  $N_j \triangleright_{\Phi} y_j\sigma$ , and
- for every  $1 \leq k \leq q$ ,  $N_{p+k} \triangleright_{\Phi} z_k\sigma$ .

Thus, by Lemma 5.3, there exists a derivation  $(\Phi, \Psi) \Rightarrow^* (\Phi_1, \Psi_1)$  using **B** rules such that  $\Psi_1 \models M|_{\alpha} \bowtie D[M_1, \dots, M_n, N_1, \dots, N_{p+q}]$ .

Besides,  $y_j\sigma$  is a subterm of some  $l_i\sigma = t_i$ . Since  $N_j \triangleright_{\Phi} y_j\sigma$ , by applying Lemma 5.2 repeatedly, we deduce that there exist some term  $M_{n+1}, \dots, M_{n+p}$  and a derivation  $(\Phi_1, \Psi_1) \Rightarrow^* (\Phi_2, \Psi_2)$  using **B** rules such that for all  $j$ ,

- $M_{n+j} \triangleright y_j\sigma$  is in  $\Phi_2$ , and
- $\Psi_2 \models M_{n+j} \bowtie N_j$ .

Let  $N = D[M_1, \dots, M_{n+p}, N_{p+1}, \dots, N_{p+q}]$ . We deduce that  $N \triangleright_{\Phi_2} l\sigma$ , and

$$\Psi_2 \models M|_{\alpha} \bowtie D[M_1, \dots, M_n, N_1, \dots, N_{p+q}] \bowtie N$$

We now consider the application to  $(\Phi_2, \Psi_2)$  of a **A** rule that involves the rewrite rule  $l \rightarrow r$ , the decomposition  $D$ , the plain terms  $(t_1, \dots, t_{n+p}) = (l_1, \dots, l_n, y_1, \dots, y_p)\sigma$ . Depending on whether  $(r\sigma) \downarrow_{\mathcal{R}}$  is ground and  $\text{Ctx}(\Phi_2^+ \vdash_{\mathcal{R}}^? r\sigma') = \perp$ , we conclude by applying **A.1**, **A.2** or **A.3**.  $\square$

### 5.3 Main theorem

We are now able to prove soundness and completeness of our transformation rules provided that a saturated state is reached.

**THEOREM 4.6 SOUNDNESS AND COMPLETENESS.** *Let  $\mathbf{E}$  be an equational theory generated by a convergent rewrite system  $\mathcal{R}$ . Let  $\varphi$  be an initial frame and  $(\Phi, \Psi)$  be a saturated state such that  $\text{Init}(\varphi) \Rightarrow^* (\Phi, \Psi)$ .*

(1) *For all  $M \in \mathcal{F}_{\text{pub}}[\text{par}(\varphi)]$  and  $t \in \mathcal{F}[\emptyset]$ , we have that:*

$$M\varphi =_{\mathbf{E}} t \iff \exists N \text{ such that } \Psi \models M \bowtie N \text{ and } N \triangleright_{\Phi} t \downarrow_{\mathcal{R}}.$$

(2) *For all  $M, N \in \mathcal{F}_{\text{pub}}[\text{par}(\varphi) \cup \mathcal{X}]$ , we have that:*

$$M\varphi =_{\mathbf{E}} N\varphi \iff \Psi \models M \bowtie N.$$

**PROOF.** Let  $\varphi$  be an initial frame and  $(\Phi, \Psi)$  be a saturated state such that  $\text{Init}(\varphi) \Rightarrow^* (\Phi, \Psi)$ .

1. ( $\Leftarrow$ ) Let  $M, N$  and  $t$  be such that  $\Psi \models M \bowtie N$  and  $N \triangleright_{\Phi} t \downarrow_{\mathcal{R}}$  (thus in particular  $N \triangleright_{\Phi}^{\mathbf{E}} t$ ). Thanks to Lemma 5.1, we have that  $M\varphi =_{\mathbf{E}} N\varphi =_{\mathbf{E}} t$ .

( $\Rightarrow$ ) Let  $M$  and  $t$  be such that  $M\varphi =_{\mathbf{E}} t$ . We have that  $M \triangleright_{\Phi} t_0 \rightarrow^* t \downarrow_{\mathcal{R}}$  for some term  $t_0$ . We show the result by induction on  $t_0$  equipped with the order  $<$  induced by the rewriting relation ( $t < t'$  if and only if  $t' \rightarrow^+ t$ ).

*Base case:*  $M \triangleright_{\Phi} t_0 = t \downarrow_{\mathcal{R}}$ . Let  $N = M$ , we have  $\Psi \models M \bowtie N$  and  $N \triangleright_{\Phi} t \downarrow_{\mathcal{R}}$ .



*Induction case:*  $M \triangleright_{\Phi} t_0 \rightarrow^+ t \downarrow_{\mathcal{R}}$ . Let  $t'$  be such that  $M \triangleright_{\Phi} t_0 \rightarrow t' \rightarrow^* t \downarrow_{\mathcal{R}}$ . Thanks to Lemma 5.4 and since  $(\Phi, \Psi)$  is already saturated<sup>1</sup>, we deduce that there exist  $N'$  and  $t''$  such that  $N' \triangleright_{\Phi} t''$ ,  $t' \rightarrow^* t''$ , and  $\Psi \models M \bowtie N'$ . We have that  $N' \triangleright_{\Phi} t'' \rightarrow^* t \downarrow_{\mathcal{R}}$  and  $t'' \leq t' < t_0$ . Thus, we can apply our induction hypothesis and we obtain that there exists  $N$  such that  $\Psi \models N' \bowtie N$  and  $N \triangleright_{\Phi} t \downarrow_{\mathcal{R}}$ .

2.( $\Leftarrow$ ) By Lemma 5.1,  $\Psi \models M \bowtie N$  implies  $M\varphi =_{\mathbf{E}} N\varphi$ .

( $\Rightarrow$ ) Let  $M$  and  $N$  such that  $M\varphi =_{\mathbf{E}} N\varphi$ . This means that there exists  $t$  such that  $M\varphi =_{\mathbf{E}} t$  and  $N\varphi =_{\mathbf{E}} t$ . By applying 1, we deduce that there exists  $M'$ ,  $N'$  such that:  $\psi \models M \bowtie M'$ ,  $M' \triangleright_{\Phi} t \downarrow_{\mathcal{R}}$ ,  $\psi \models N \bowtie N'$  and  $N' \triangleright_{\Phi} t \downarrow_{\mathcal{R}}$ . Thanks to Lemma 5.3 and since  $(\Phi, \Psi)$  is already saturated, we easily deduce that  $\Psi \models M' \bowtie N'$ , and thus  $\Psi \models M \bowtie N$ .  $\square$

We proved that saturated frames yield sound and complete characterizations of deducible terms and visible equations of their initial frames. Yet, the saturation procedure may still not terminate, or fail due to rule **A.3**.

## 6. NON-FAILURE

As shown by the following example (from [Ciobăcă et al. 2009]), our procedure may fail.

*Example 6.1.* Consider the theory  $\mathbf{E}_{\text{mal}}$  given below:

$$\mathbf{E}_{\text{mal}} = \{\text{dec}(\text{enc}(x, y), y) = x, \text{mal}(\text{enc}(x, y), z) = \text{enc}(z, y)\}.$$

The **mal** function symbol allows one to arbitrarily change the plaintext of an encryption. Such a malleable encryption is not realistic. It is only used for illustrative purpose.

By orienting from left to right the equations, we obtain a convergent rewrite system. Thus,  $\mathbf{E}_{\text{mal}}$  is a convergent equational theory. Let  $\varphi = \{\mathbf{w}_1 \triangleright \text{enc}(s, \mathbf{k})\}$  where  $s$  and  $\mathbf{k}$  are private constants. The only rule that is applicable is an instance of an **A** rule. Consider the rewrite rule  $\text{mal}(\text{enc}(x, y), z) \rightarrow \text{enc}(z, y)$  and the only deduction fact in  $\text{Init}(\varphi) = (\varphi, \emptyset)$ . We obtain  $r\sigma \downarrow_{\mathcal{R}} = \text{enc}(z, \mathbf{k})$ . This term is not ground and the condition required in case (1) is not fulfilled. Thus, we have that  $\text{Init}(\varphi) \Rightarrow \perp$ . Note that, since no other rule is applicable, there is no hope to find a strategy of rule applications to handle this case.

In this section, we identify a class of theories, called *layered convergent* theories, (a syntactically defined class of theories) for which failure is guaranteed not to occur.

### 6.1 Layered convergent theories

We prove that the algorithm never fails for *layered convergent* theories. Layered convergent theories consist in a generalization of subterm theories, considering each decomposition of the rewrite rules of the theory.

*Definition 6.2 layered rewrite system.* A rewrite system  $\mathcal{R}$ , and by extension its equational theory  $\mathbf{E}$ , are *layered* if there exists an ascending chain of subsets  $\emptyset = \mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \dots \subseteq \mathcal{R}_{N+1} = \mathcal{R}$  ( $N \geq 0$ ), such that for every  $0 \leq$

<sup>1</sup>Note that rule **A.3** is never applicable on a saturated state.

$i \leq N$ , for every rule  $l \rightarrow r$  in  $\mathcal{R}_{i+1} - \mathcal{R}_i$ , for every  $(n, p, q)$ -decomposition  $l = D[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$ , one of the following two conditions holds:

- (i)  $\text{var}(r) \subseteq \text{var}(l_1, \dots, l_n)$ ;
- (ii) there exist  $C_0, C_1, \dots, C_k$  and  $s_1, \dots, s_k$  such that
  - $r = C_0[s_1, \dots, s_k]$ ;
  - for each  $1 \leq i \leq k$ ,  $C_i[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$  rewrites to  $s_i$  in zero or one step of rewrite rule in head position along  $\mathcal{R}_i$ .

In the latter case, we say that the context  $C = C_0[C_1, \dots, C_k]$  is *associated* to the decomposition  $D$  of  $l \rightarrow r$ . Note that  $C[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q] \rightarrow_{\mathcal{R}_i}^* r$ .

The large class of weakly subterm convergent is an (easy) particular case of layered convergent theories.

LEMMA 6.3. *Any weakly subterm convergent rewrite system  $\mathcal{R}$  is layered convergent.*

PROOF. Let  $N = 0$  and  $\mathcal{R}_1 = \mathcal{R}$ . For any  $l \rightarrow r$  in  $\mathcal{R}$  and for every decomposition  $l = D[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$ , the term  $r$  is a subterm of  $l$ , thus either  $r = C[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$  for some context  $C$ , or  $r$  is a subterm of some  $l_i$  thus  $\text{var}(r) \subseteq \text{var}(l_1, \dots, l_n)$ .  $\square$

Consider the convergent theories of blind signatures  $\mathbf{E}_{\text{blind}}$  and prefix encryption  $\mathbf{E}_{\text{pref}}$  defined by the following sets of equations.

$$\mathcal{E}_{\text{blind}} = \left\{ \begin{array}{l} \text{checksign}(\text{sign}(x, y), \text{pub}(y)) = \text{ok} \\ \text{unblind}(\text{blind}(x, y), y) = x \\ \text{unblind}(\text{sign}(\text{blind}(x, y), z), y) = \text{sign}(x, z) \end{array} \right\}$$

$$\mathcal{E}_{\text{pref}} = \mathcal{E}_{\text{enc}} \cup \{ \text{pref}(\text{enc}(\langle x, y \rangle, z)) = \text{enc}(x, z) \}$$

The theory  $\mathbf{E}_{\text{blind}}$  models primitives used in e-voting protocols [Delaune et al. 2009]. The prefix theory represents the property of many chained modes of encryption (e.g. CBC) where an attacker can retrieve any encrypted prefix out of a ciphertext.

LEMMA 6.4. *The rewrite system associated to the theory of homomorphism  $\mathbf{E}_{\text{hom}}$  defined in Section 2.3 as well as the rewrite systems obtained by orienting from left to right the equations in  $\mathbf{E}_{\text{blind}}$  and  $\mathbf{E}_{\text{pref}}$  are layered convergent.*

PROOF. Let us check for instance that the prefix theory  $\mathbf{E}_{\text{pref}}$  is layered. Let  $N = 1$ ,  $\mathcal{R}_1$  be the rewrite system obtained from  $\mathcal{E}_{\text{enc}}$  by orienting the equations from left to right, and  $\mathcal{R}_2 = \mathcal{R}_1 \cup \{ \text{pref}(\text{enc}(\langle x, y \rangle, z)) \rightarrow \text{enc}(x, z) \}$ . The rewrite rules of  $\mathcal{R}_1$  satisfy the assumptions since  $\mathcal{R}_1$  forms a convergent subterm rewrite system. The additional rule  $\text{pref}(\text{enc}(\langle x, y \rangle, z)) \rightarrow \text{enc}(x, z)$  admits three decompositions up to permutation of parameters:

- $l = \text{pref}(l_1)$ , in which case  $\text{var}(r) \subseteq \text{var}(l_1)$ ;
- $l = \text{pref}(\text{enc}(l_1, z))$ , in which case  $\text{enc}(\text{proj}_1(l_1), z) \rightarrow_{\mathcal{R}_1} r$ ;
- $l = \text{pref}(\text{enc}(\langle x, y \rangle, z))$ , in which case  $r = \text{enc}(x, z)$ .

Verifying that the convergent theories  $\mathbf{E}_{\text{hom}}$  and  $\mathbf{E}_{\text{blind}}$  are layered is similar.  $\square$

## 6.2 A syntactic criterion

*Definition 6.5 Maximal.* We say that the function  $\text{Ctx}$  is *maximal* if for every  $\phi$  and  $t$ , if there exists  $s$  such that  $\phi \vdash s$  and  $t \rightarrow_{\mathcal{R}}^* s$ , then  $\text{Ctx}(\phi \vdash_{\mathcal{R}}^? t) \neq \perp$ .

**PROPOSITION 6.6.** *Assume that the function  $\text{Ctx}$  in use is maximal. Then, provided that  $\mathcal{R}$  is layered convergent, there exists no state  $(\Phi, \Psi)$  from which  $(\Phi, \Psi) \Rightarrow \perp$  is the only applicable derivation.*

**PROOF.** By contradiction, let  $(\Phi, \Psi)$  be a state from which  $(\Phi, \Psi) \Rightarrow \perp$  is the only applicable derivation, and let  $l \rightarrow r$  be the rewrite rule involved in the corresponding instance of **A.3**. We prove the property by induction on the index  $i \in \{0 \dots N\}$  such that  $l \rightarrow r \in \mathcal{R}_{i+1} - \mathcal{R}_i$ . Using the notations of Figure 1 for the instance of **A.3** under consideration and the assumption on  $\text{Ctx}$ , we have that:

- (a) for every  $r\sigma \rightarrow_{\mathcal{R}}^* s$ ,  $\Phi \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\} \not\vdash s$ , and
- (b)  $(r\sigma)\downarrow_{\mathcal{R}}$  is not ground.

In particular, (b) implies that  $\text{var}(r)$  is not included in  $\text{var}(l_1, \dots, l_n)$ , otherwise we would have

$$\begin{aligned} \text{var}((r\sigma)\downarrow_{\mathcal{R}}) &\subseteq \text{var}(r\sigma) \subseteq \text{var}(\text{var}(r)\sigma) \\ &\subseteq \text{var}(\text{var}(l_1, \dots, l_n)\sigma) \subseteq \text{var}(t_1, \dots, t_n) = \emptyset \end{aligned}$$

By assumption on the decomposition  $l = D[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$  of  $l \rightarrow r \in \mathcal{R}_{i+1} - \mathcal{R}_i$ , we deduce that there exists some contexts  $C_0, \dots, C_k$  and some terms  $s_1, \dots, s_k$  such that:

- $r = C_0[s_1, \dots, s_k]$ ;
- for each  $1 \leq i \leq k$ ,  $C_i[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$  rewrites to  $s_i$  in zero or one step of rewrite rule in head position along  $\mathcal{R}_i$ .

Let  $C = C_0[C_1, \dots, C_k]$  and  $t_0 = C[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$ . Note that  $t_0 \rightarrow_{\mathcal{R}_i}^* r$ . If  $t_0 = r$ , we obtain that  $r\sigma = C[t_1, \dots, t_{n+p}, z_1, \dots, z_q]$  is syntactically deducible from  $\Phi \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\}$ , which contradicts (a). Hence  $t_0 \rightarrow_{\mathcal{R}_i}^+ r$ , and in particular  $i > 0$ .

Let  $\mu$  be a substitution mapping the variables  $z_j$  to distinct fresh public constants  $\mathbf{a}_j$ . For each  $1 \leq i \leq k$ , let  $u_i = C_i[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]\sigma\mu$ . The term  $u_i = C_i[t_1, \dots, t_{n+p}, \mathbf{a}_1, \dots, \mathbf{a}_q]$  is syntactically deducible from  $\Phi$ , and reduces to  $u'_i = s_i\sigma\mu$  in zero or one step (in head position) along  $\mathcal{R}_i$ .

By induction hypothesis on  $i - 1$ , no applicable rule **A.3** from  $(\Phi, \Psi)$  may involve a rule in  $\mathcal{R}_i$ . Besides, by assumption,  $(\Phi, \Psi)$  is saturated for the rules **B.1**, **B.2**, **A.1** and **A.2**. Therefore, Lemma 5.4 applied to  $\Phi \vdash u_i$  and  $u_i \rightarrow_{\mathcal{R}_i} u'_i$  implies that there exists  $u''_i$  such that  $u'_i \rightarrow_{\mathcal{R}}^* u''_i$  and  $\Phi \vdash u''_i$ . The same conclusion trivially holds if  $u'_i = u_i$ . Let  $s = C_0[u''_1, \dots, u''_k]\mu^{-1}$  be the term obtained by replacing each  $\mathbf{a}_i$  by  $z_i$  in  $C[u''_1, \dots, u''_k]$ . Since the  $\mathbf{a}_i$  do not occur in  $\mathcal{R}$  nor in  $\Phi$ , we deduce that  $s$  satisfies  $r\sigma = C_0[s_1\sigma, \dots, s_k\sigma] = C_0[u''_1, \dots, u''_k]\mu^{-1} \rightarrow_{\mathcal{R}}^* s$  and  $\Phi \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\} \vdash s$ , in contradiction with the condition (a) stated at the beginning of the proof.  $\square$

### 6.3 Practical considerations.

Unfortunately, such a maximal Ctx is too inefficient in practice as one has to consider the syntactic deducibility problem  $\phi \vdash s$  for every  $t \rightarrow_{\mathcal{R}}^* s$ . Proposition 6.7 below shows that the simple function context is actually sufficient to ensure non-failure when we know that another function Ctx already prevents failure on any state (reachable or not).

**PROPOSITION 6.7.** *Let  $\mathcal{R}$  be a convergent rewrite system and  $\text{Ctx}_0$  be an arbitrary function Ctx. If there exists no state  $(\Phi, \Psi)$  from which  $(\Phi, \Psi) \Rightarrow \perp$  is the only applicable derivation when the function Ctx in use is  $\text{Ctx}_0$ , then there exists no state  $(\Phi, \Psi)$  from which  $(\Phi, \Psi) \Rightarrow \perp$  is the only applicable derivation for any choice of Ctx.*

**PROOF.** Let  $\text{Ctx}_0$  and  $\text{Ctx}'_0$  be two arbitrary functions Ctx (i.e. they satisfy properties (a) and (b)). Assume that there exists no state  $(\Phi, \Psi)$  from which  $(\Phi, \Psi) \Rightarrow \perp$  is the only applicable derivation when the function Ctx in use is  $\text{Ctx}_0$ . Assume by contradiction that there exists a state  $(\Phi_0, \Psi_0)$  from which  $(\Phi_0, \Psi_0) \Rightarrow \perp$  is the only applicable derivation for  $\text{Ctx}'_0$ . This means that there exist:

- a rewrite rule  $l \rightarrow r \in \mathcal{R}$ ,
- a proper decomposition  $D[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$  of  $l$ ,
- some deduction facts  $M_1 \triangleright t_1, \dots, M_{n+p} \triangleright t_{n+p} \in \Phi_0$ , and
- a substitution  $\sigma$  such that  $(l_1, \dots, l_n, y_1, \dots, y_p)\sigma = (t_1, \dots, t_{n+p})$ .

Moreover, since this instance corresponds to an instance of **A.3**, we have that  $r\sigma \downarrow_{\mathcal{R}}$  is not ground. When the function Ctx in use is  $\text{Ctx}_0$ , this instance has to correspond to an instance of **A.1** (**A.2** and **A.3** are impossible). Hence, we have that  $\text{Ctx}_0(\Phi_0 \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\}) \vdash_{\mathcal{R}}^? r\sigma \neq \perp$ . This means that there exists  $s$  such that  $r\sigma \rightarrow_{\mathcal{R}}^* s$  and  $\Phi_0 \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\} \vdash s$ . Since  $\mathcal{R}$  is convergent, we have that  $s \rightarrow_{\mathcal{R}}^* r\sigma \downarrow_{\mathcal{R}}$ .

Let  $\mu$  be a substitution mapping the variables  $z_j$  to distinct fresh public constants  $a_j$ . We have that  $s\mu \rightarrow_{\mathcal{R}}^* (r\sigma \downarrow_{\mathcal{R}})\mu$  and also that  $\Phi_0 \vdash s\mu$ . Since  $(\Phi_0, \Psi_0) \Rightarrow \perp$  is the only applicable derivation for  $\text{Ctx}'_0$ , the rules **A.2**, **B.1**, and **B.2** cannot be applicable, even for  $\text{Ctx}_0$ . We saturate  $(\Phi_0, \Psi_0)$  with the **A.1** rule for  $\text{Ctx}_0$ , reaching a state of the form  $(\Phi_0, \Psi'_0)$  since only equations can be added to the state. Note also that the **A.1** rule can only be applied a finite a number of time and does not trigger the other rules. Thus  $(\Phi_0, \Psi'_0)$  is saturated for  $\text{Ctx}_0$ . Using Lemma 5.4 (with the function  $\text{Ctx}_0$ ), we obtain that  $\Phi_0 \triangleright (r\sigma \downarrow_{\mathcal{R}})\mu$ , and thus  $\Phi_0 \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\} \vdash r\sigma \downarrow_{\mathcal{R}}$ . This contradicts the fact that **A.1** does not apply on  $(\Phi_0, \Psi_0)$  when the function Ctx in use is  $\text{Ctx}'_0$ . Hence, the result.  $\square$

**COROLLARY 6.8.** *Let  $\mathcal{R}$  be a layered convergent rewrite system and consider an arbitrary function Ctx in use. There exists no state  $(\Phi, \Psi)$  from which  $(\Phi, \Psi) \Rightarrow \perp$  is the only applicable derivation.*

## 7. TERMINATION

In the previous section, we have described a sufficient criterion for non-failure. As shown by the example given below, this criterion does not ensure the termination of our saturation procedure.

*Example 7.1.* Consider the following layered convergent rewrite system  $f(g(x)) \rightarrow g(h(x))$  where  $f$  is a public function symbol whereas  $g$  and  $h$  are private function symbols. Let  $\varphi = \{w_0 \triangleright g(a)\}$  where  $a$  is a private constant. By repeatedly applying the **A** rule on the newly generated deduction fact, we generate an infinite number of deduction facts of the form:

$$f(w_0) \triangleright g(h(a)), f(f(w_0)) \triangleright g(h(h(a))), f(f(f(w_0))) \triangleright g(h(h(h(a)))), \dots$$

To obtain decidability for a given layered convergent theory, there remains only to provide a termination argument. Such an argument is generally easy to develop by hand as we illustrate on the example of the prefix theory. For the case of existing decidability results from [Abadi and Cortier 2006], such as the theories of blind signature and homomorphic encryption, we also provide a semantic criterion that allows us to directly conclude termination of the procedure. Note that this semantic criterion does not apply only to layered convergent theories but to any convergent theories (for which failure is guaranteed not to happen).

### 7.1 Termination of **B** rules

To begin with, we note that **B** rules always terminate after a polynomial number of steps. Let us write  $\dot{\Rightarrow}^n$  for the relation made of exactly  $n$  *strict applications* of rules ( $S \dot{\Rightarrow} S'$  iff  $S \Rightarrow S'$  and  $S \neq S'$ ).

**PROPOSITION 7.2.** *For every states  $S = (\Phi, \Psi)$  and  $S'$  such that  $S \dot{\Rightarrow}^n S'$  using only **B** rules,  $n$  is polynomially bounded in the size of  $\text{im}(\Phi)$ .*

This is due to the fact that frames are one-to-one and that the rule **B.2** only adds deduction facts  $M \triangleright t$  such that  $t$  is a subterm of an existing term in  $\Phi$ .

### 7.2 Proving termination by hand.

For proving termination, we observe that it is sufficient to provide a function  $s$  mapping each frame  $\Phi$  to a finite set of terms  $s(\Phi)$  including the subterms of  $\text{im}(\Phi)$  and such that rule **A.2** only adds deduction facts  $M \triangleright t$  satisfying  $t \in s(\Phi)$ .

For subterm theories, we obtain polynomial termination by choosing  $s(\Phi)$  to be the subterms of  $\text{im}(\Phi)$  together with the ground right-hand sides of  $\mathcal{R}$ .

**PROPOSITION 7.3.** *Let  $\mathbf{E}$  be a weakly subterm convergent theory. For every  $S = (\Phi, \Psi)$  and  $S'$  such that  $S \dot{\Rightarrow}^n S'$ ,  $n$  is polynomially bounded in the size of  $\text{im}(\Phi)$ .*

To conclude that deduction and static equivalence are decidable in polynomial time [Abadi and Cortier 2006], we need to show that the deduction facts and the equations are of polynomial size. This requires a DAG representation for terms and visible equations. For our implementation, we have chosen not to use DAGs for the sake of simplicity since DAGs require much heavier data structures. However, similar techniques as those described in [Abadi and Cortier 2006] would apply to implement our procedure using DAGs.

For proving termination for the prefix theory  $\mathbf{E}_{\text{pref}}$ , it suffices to consider  $s(\phi) = \text{st}_{\text{ext}}(\Phi)$ , where the notion of extended subterm is recursively defined as follows:

$$\text{—st}_{\text{ext}}(a) = \{a\} \quad \text{if } a \text{ is a constant or a variable}$$

$$\begin{aligned}
& \text{—st}_{\text{ext}}(f(t_1, \dots, t_n)) = \{f(t_1, \dots, t_n)\} \cup \bigcup_{i=1}^n \text{st}_{\text{ext}}(t_i) \quad f \in \{\text{dec}, \langle, \rangle, \text{proj}_1, \text{proj}_2, \text{pref}\} \\
& \text{—st}_{\text{ext}}(\text{enc}(t, u)) = \{\text{enc}(t, u), \text{enc}(t_1, u)\} \cup \text{st}_{\text{ext}}(t) \cup \text{st}_{\text{ext}}(u) \quad \text{if } t = \langle t_1, t_2 \rangle \\
& \text{—st}_{\text{ext}}(\text{enc}(t, u)) = \{\text{enc}(t, u)\} \cup \text{st}_{\text{ext}}(t) \cup \text{st}_{\text{ext}}(u) \quad \text{otherwise.}
\end{aligned}$$

PROPOSITION 7.4. *Consider the prefix theory  $\mathbf{E}_{\text{pref}}$ . For every  $S = (\Phi, \Psi)$  and  $S'$  such that  $S \xRightarrow{\bullet}^n S'$ ,  $n$  is polynomially bounded in the size of  $\text{im}(\Phi)$ .*

We then deduce that deduction and static equivalence are decidable for the equational theory  $\mathbf{E}_{\text{pref}}$ , which is a new decidability result.

COROLLARY 7.5. *Deduction and static equivalence are decidable in polynomial time for the equational theory  $\mathbf{E}_{\text{pref}}$ .*

Similarly, we may retrieve decidability of deduction and static equivalence for  $\mathbf{E}_{\text{hom}}$  and  $\mathbf{E}_{\text{blind}}$ . However, we provide another criterion that allows one to derive these facts from existing results.

### 7.3 A semantic criterion

We now provide a semantic criterion that more generally explains why our procedure succeeds on theories previously known to be decidable [Abadi and Cortier 2006]. This criterion intuitively states that the set of deducible terms from any initial frame  $\varphi$  should be equivalent to a set of *syntactically* deducible terms. Provided that failures are prevented and assuming a *fair* strategy for rule application, we prove that this criterion is a necessary and sufficient condition for our procedure to terminate.

*Definition 7.6 Fair derivation.* An infinite derivation

$$(\Phi_0, \Psi_0) \Longrightarrow \dots \Longrightarrow (\Phi_n, \Psi_n) \Longrightarrow \dots$$

is *fair* iff along this derivation,

- (a) **B** rules are applied with greatest priority, and
- (b) whenever a **A** rule is applicable for some instance  $(l \rightarrow r, D, t_1, \dots, t_n, \dots)$ , eventually the same instance of rule is applied during the derivation.

Fairness implies that any deducible term is eventually syntactically deducible. This result follows from Lemma 5.3 and Lemma 5.4.

LEMMA 7.7. *Let  $S_0 = (\Phi_0, \Psi_0) \Longrightarrow \dots \Longrightarrow (\Phi_n, \Psi_n) \Longrightarrow \dots$  be an infinite fair derivation from a state  $S_0$ . For every ground term  $t$  such that  $\Phi_0 \vdash_{\mathbf{E}} t$ , either  $(\Phi_0, \Psi_0) \Longrightarrow^* \perp$  or there exists  $i$  such that  $\Phi_i \vdash t \downarrow_{\mathcal{R}}$ .*

PROOF. Let  $t$  be a ground term deducible from  $\Phi_i$  modulo  $\mathbf{E}$ . There exists  $t_0$  such that  $M \triangleright_{\Phi_i} t_0$  and  $t_0 \rightarrow^* t \downarrow_{\mathcal{R}}$ . This means that there exist a (public) context  $C$  and some deduction facts  $M_1 \triangleright t_1, \dots, M_n \triangleright t_n \in \Phi_i$  such that  $M = C[M_1, \dots, M_n]$  and  $t_0 = C[t_1, \dots, t_n]$ .

We show that either  $(\Phi_i, \Psi_i) \Longrightarrow^* \perp$  or there exists  $j \geq i$  such that  $t \downarrow_{\mathcal{R}}$  is syntactically deducible from  $\Phi_j$ , by induction on  $t_0$  equipped with the order  $<$  induced by the rewriting relation (that is  $t_1 < t_2$  if and only if  $t_2 \rightarrow^+ t_1$ ).

*Base case:*  $t_0 = t \downarrow_{\mathcal{R}}$ . In such a case, since  $\Phi_i \vdash t_0$ , we have that  $\Phi_i \vdash t \downarrow_{\mathcal{R}}$ . This allows us to conclude.

*Induction step:*  $t_0 \rightarrow t' \rightarrow^* t \downarrow_{\mathcal{R}}$ .

Along a fair derivation, **B** rules are applied in priority. Hence, we choose the smallest  $i_1 \geq i$  such that no more **B** rules can be applied from  $(\Phi_{i_1}, \Psi_{i_1})$ . Note indeed that there is no infinite derivation with only **B** rules (Proposition 7.2). We have still that  $C[M_1, \dots, M_n] \triangleright_{\Phi_{i_1}} t_0 \rightarrow t'$ .

Applying Lemma 5.4 and observing that no **B** rule can be applied from  $(\Phi_{i_1}, \Psi_{i_1})$ , we are in one of the following cases:

- $(\Phi_{i_1}, \Psi_{i_1}) \Longrightarrow \perp$ . In such a case, we easily conclude since  $(\Phi_0, \Psi_0) \Longrightarrow^* \perp$ .
- $\Phi_{i_1} \vdash t''$  for some  $t''$  such that  $t' \rightarrow_{\mathcal{R}}^* t''$ . In such a case, we conclude by applying our induction hypothesis since  $t'' < t' < t_0$ . There exists  $j \geq i_1$  such that  $\Phi_j \vdash t \downarrow_{\mathcal{R}}$ .
- Otherwise an instance  $(l \rightarrow r, D, t_1, \dots, t_n, \dots)$  of a **A** rule is applicable. Note that this instance is entirely determined by the rewrite rule  $l \rightarrow r$  involved in the rewriting step  $t_0 \rightarrow t'$ , the deduction facts  $M_i \triangleright t_i$  ( $1 \leq i \leq n$ ) and the public context that witness the fact that  $\Phi_i \vdash t_0$ .

By fairness, we know that a **A** rule will be applied along the derivation for the same instance  $(l \rightarrow r, D, t_1, \dots, t_n, \dots)$ . Let  $i_2$  be the indice on which this instance is applied. We have that  $i_2 \geq i_1$ . Note that since **B** rules are applied in priority,  $(\Phi_{i_2}, \Psi_{i_2})$  is saturated for **B** rules. Either, we have that  $(\Phi_{i_2}, \Psi_{i_2}) \Longrightarrow \perp$  (and thus  $(\Phi_i, \Psi_i) \Longrightarrow^* \perp$ ) or  $(\Phi_{i_2}, \Psi_{i_2}) \Longrightarrow (\Phi_{i_2+1}, \Psi_{i_2+1})$ .

We have that  $C[M_1, \dots, M_n] \triangleright_{\Phi_{i_2}} t_0$  and  $t_0 \rightarrow_{\mathcal{R}} t'$ . By Lemma 5.4, either  $(\Phi_{i_2}, \Psi_{i_2}) \Longrightarrow \perp$  or there exists  $(\Phi'_{i_2}, \Psi'_{i_2})$ ,  $M'$  and  $t''$  such that:

- $(\Phi_{i_2}, \Psi_{i_2}) \Longrightarrow (\Phi'_{i_2}, \Psi'_{i_2})$ ;
- $M' \triangleright_{\Phi'_{i_2}} t''$  with  $t' \rightarrow_{\mathcal{R}}^* t''$ ; and
- $\Psi'_{i_2} \models C[M_1, \dots, M_n] \bowtie M'$ .

Actually, the instance of the **A** rule that is applied in this derivation is entirely determined by the rewrite rule  $l \rightarrow r$  involved in the rewriting step  $t_0 \rightarrow t'$ , the public context  $C$  and the deduction facts  $M_i \triangleright t_i$  ( $1 \leq i \leq n$ ) that witness the fact that  $\Phi_i \vdash t_0$  (and thus  $\Phi_{i_2} \vdash t_0$ ). Hence, we have that  $(\Phi'_{i_2}, \Psi'_{i_2}) = (\Phi_{i_2+1}, \Psi_{i_2+1})$ .

Thus we have that  $M' \triangleright_{\Phi'_{i_2+1}} t''$  with  $t'' \rightarrow^* t \downarrow_{\mathcal{R}}$  and  $t'' < t' < t$ . We can apply our induction hypothesis, either  $(\Phi_{i_2+1}, \Psi_{i_2+1}) \Longrightarrow^* \perp$  (and thus  $(\Phi_i, \Psi_i) \Longrightarrow^* \perp$ ) or there exists  $j \geq i_2 + 1$  such that  $\Phi_j \vdash t \downarrow_{\mathcal{R}}$ .  $\square$

Our termination criteria (Property (ii) below) is a semantic criterion. It is related to the notion *locally stable* introduced in [Abadi and Cortier 2006].

**PROPOSITION 7.8 CRITERION FOR TERMINATION.** *Let  $\varphi$  be an initial frame such that  $\text{Init}(\varphi) \not\Rightarrow^* \perp$ . The following conditions are equivalent:*

- (i) *There exists a saturated couple  $(\Phi, \Psi)$  such that  $\text{Init}(\varphi) \Longrightarrow^* (\Phi, \Psi)$ .*
- (ii) *There exists a (finite) initial frame  $\varphi_s$  such that for every term  $t$ ,  $t$  is deducible from  $\varphi$  modulo  $E$  iff  $t \downarrow_{\mathcal{R}}$  is syntactically deducible from  $\varphi_s$ .*
- (iii) *There exists no fair infinite derivation starting from  $\text{Init}(\varphi)$ .*



PROOF.  $(iii) \Rightarrow (i)$ : trivial. Indeed by using a fair derivation we will eventually reach a weakly saturated state.  $(i) \Rightarrow (ii)$ : Let  $\Phi = \{M_1 \triangleright s_1, \dots, M_\ell \triangleright s_\ell\}$  and  $\varphi_s = \{w_1 \triangleright s_1, \dots, w_\ell \triangleright s_\ell\}$ . Let  $t$  be a ground term. By Theorem 4.6, we have that  $\exists M.M \triangleright_\varphi^E t$  iff  $\exists M.M \triangleright_\Phi t \downarrow_{\mathcal{R}}$ , i.e.  $\exists M.M \triangleright_{\varphi_s} t \downarrow_{\mathcal{R}}$ .  $(ii) \Rightarrow (iii)$ : we need to prove that there exists no fair infinite derivation starting from  $\text{Init}(\varphi)$ .

Let  $\varphi_s = \{w_1 \triangleright s_1, \dots, w_\ell \triangleright s_\ell\}$  an initial frame such that for every  $t$ ,  $\exists M.M \triangleright_\varphi^E t$  is equivalent to  $\exists M.M \triangleright_{\varphi_s} t \downarrow_{\mathcal{R}}$ . Assume by contradiction that there is an infinite fair derivation  $(\Phi_0, \Psi_0) \Rightarrow \dots \Rightarrow (\Phi_n, \Psi_n) \Rightarrow \dots$  with  $(\Phi_0, \Psi_0) = \text{Init}(\varphi)$ .

By Lemma 7.7 and since  $\text{Init}(\varphi) \not\Rightarrow^* \perp$ , we deduce that there exists  $i_0$  such that each  $s_i$ ,  $1 \leq i \leq \ell$  is syntactically deducible from  $\Phi_{i_0}$ . Since there is no infinite derivation with only **B** rules (Proposition 7.2), we can also assume that no **B** rule can be applied from  $\Phi_{i_0}$ . We have that  $\exists M.M \triangleright_\varphi^E t$  is now equivalent to  $\exists M.M \triangleright_{\Phi_{i_0}} t \downarrow_{\mathcal{R}}$  thus the **A.2** rule cannot be applied either. We deduce that no deduction facts are added to  $\Phi_{i_0}$  along the derivation, that is  $\Phi_j = \Phi_{i_0}$  for every  $j \geq i_0$ . Since no deduction fact are added, only a finite number of **A.1** rules can be applied, which contradicts the existence of an infinite chain.  $\square$

Together with the syntactic criterion described in Section 6 to prevent non-failure, this criterion (Property  $(ii)$ ) allows us to prove decidability of deduction and static equivalence for layered convergent theories that belong to the class of locally stable theories defined in [Abadi and Cortier 2006]. As a consequence, our procedure always saturates for the theories of blind signatures and homomorphic encryption since those theories are layered and have been proved locally stable [Abadi and Cortier 2006]. Other examples of layered convergent theories enjoying this criterion can be found in [Abadi and Cortier 2006] (e.g. a theory of addition). While in [Abadi and Cortier 2006] the decision algorithm needs to be adapted for each theory, we propose a single (and efficient) algorithm that ensures a unified treatment of all these theories.

## 8. IMPLEMENTATION: THE TOOL YAPA

YAPA (Yet Another Protocol Analyzer) is an Ocaml implementation of the saturation procedure presented in Section 4 with several optional optimizations. It can be freely downloaded<sup>2</sup> together with a brief manual and examples.

The tool takes as input an equational theory described by a finite convergent rewrite system, as well as frame definitions and queries. The procedure starts by computing the decompositions of the rewrite system. By default, the following optimization is done: provided that the rewrite rules are given in an order compatible with the sets  $\mathcal{R}_0 \subseteq \dots \subseteq \mathcal{R}_{N+1}$  of Definition 6.2, the tool is able to recognize layered theories and to pre-compute the associated contexts  $C$  related to condition (ii) of this definition. This allows resolving the failure cases as soon as they appear, rather than later on, when the saturation procedure has made enough progress. This optimization was studied in a first version of this article [Baudet et al. 2009] but as the practical benefits appear to be minor (see below), we chose not to keep these technical developments in this version for the sake of notational simplicity.

<sup>2</sup><http://www.lsv.ens-cachan.fr/~baudet/yapa/index.html>

Another optimization concerns a specific treatment of subterm convergent theories but does not induce any difference with the theoretical procedure presented here. Except for the first (optional) optimization mentioned above, the algorithm follows the procedure described in Section 4, using a minimal function  $\text{Ctx}$  in the sense in Section 6.3, and a fair strategy of rule application (see Definition 7.6).

We have conducted several experiments on a PC Intel Core 2 Duo at 2.4 GHz with 2 Go RAM for various equational theories (see below) and found that YAPA provides an efficient way to check static equivalence and deducibility. Those examples are available at <http://www.lsv.ens-cachan.fr/~baudet/yapa/index.html>. The figures given below are valid for the versions with and without optimizations.

For the case of  $E_{\text{enc}}$ , we have run YAPA on the frames:

$$\begin{aligned} -\varphi_n &= \{w_1 \triangleright t_n^0, w_2 \triangleright c_0, w_3 \triangleright c_1\}, \text{ and} \\ -\varphi'_n &= \{w_1 \triangleright t_n^1, w_2 \triangleright c_0, w_3 \triangleright c_1\}, \end{aligned}$$

where  $t_0^i = c_i$  and  $t_{n+1}^i = \langle \text{enc}(t_n^i, k_n^i), k_n^i \rangle$ ,  $i \in \{0, 1\}$ . These examples allow us to increase the (tree, non-DAG) size of the distinguishing tests exponentially, while the sizes of the frames grow linearly. Despite the size of the output, we have observed satisfactory performances for the tool.

Equational theory	$E_{\text{enc}}$ $n = 10$	$E_{\text{enc}}$ $n = 14$	$E_{\text{enc}}$ $n = 16$	$E_{\text{enc}}$ $n = 18$	$E_{\text{enc}}$ $n = 20$
Execution time	< 1s	1,7s	8s	30s	< 3min

We have also experimented YAPA on several convergent theories, e.g.  $E_{\text{blind}}$ ,  $E_{\text{hom}}$ ,  $E_{\text{pref}}$  and the theory of addition  $E_{\text{add}}$  defined in [Abadi and Cortier 2006].

*Comparison with ProVerif.* In comparison with the tool ProVerif [Blanchet 2001; Blanchet et al. 2008], here instrumented to check static equivalences, our test samples suggest a running time between one and two orders of magnitude faster for YAPA. Also we did not succeed in making ProVerif terminate on the two theories  $E_{\text{hom}}$  and  $E_{\text{add}}$ . Of course, these results are not entirely surprising given that ProVerif is tailored for the more general (and difficult) problem of protocol (in)security under active adversaries. In particular ProVerif's initial preprocessing of the rewrite system appears more substantial than ours and does not terminate on the theories  $E_{\text{hom}}$  and  $E_{\text{add}}$  (although termination is guaranteed for linear or subterm-convergent theories [Blanchet et al. 2008]).

*Comparison with KiSs.* The tool KiSs (Knowledge in Security protocols) is a C++ implementation of the procedure described in [Ciobăcă et al. 2009]. This procedure reused the same concepts than the one presented in a preliminary version of this work [Baudet et al. 2009]. The performances of the tool YAPA are comparable to the performances of KiSs. However, since the tool KiSs implements DAG representations for terms, it does better on the example developed above. From the point of view of the equational theories the tools are able to deal with, they are incomparable. KiSs allows one to consider some equational theories for which our procedure fails (e.g. the theory of trapdoor bit commitment).

Conversely our procedure is guaranteed to terminate (without failure) for theories that are not considered by the procedure implemented in KiSs. The only general

class of theory for which KiSS has been proved to terminate is the class of subterm convergent equational theory.

## 9. CONCLUSION AND FUTURE WORK

We have proposed a procedure for checking deducibility and static equivalence. Our procedure is correct and complete for any convergent theory and is efficient, as shown by its implementation within the tool YAPA. Since deducibility and static equivalence are undecidable in general, our algorithm may fail or may not terminate. We have identified a large class of equational theories (called layered convergent) for which non-failure of the procedure is ensured. Since termination can then often be easily proved by hand, we have obtained a new decidability result for the prefix theory. We have also proposed a semantic (and exact) characterization for the procedure to terminate. This again yields a new decidability result for locally stable, layered convergent theories.

As further work, we would like to extend our procedure to theories with associative and commutative operators. A first possibility would be to implement the decidability result of [Cortier and Delaune 2007] for monoidal theories (that include many theories with associative and commutative operators) and to combine the two procedures using the combination theorem of [Arnaud et al. 2007]. However, it seems much more efficient to integrated associativity and commutativity directly and this could even open the way to a more powerful combination technique.

The tool KiSS, developed recently [Ciobâcă et al. 2009], supports several equational theories for which our procedure fails. Conversely our procedure is guaranteed to terminate (without failure) for classes of theories that are not considered by the procedure implemented in KiSS. It would be interesting to compare the techniques and possibly to combine them in order to capture more theories.

## REFERENCES

- ABADI, M., BAUDET, M., AND WARINSCHI, B. 2006. Guessing attacks and the computational soundness of static equivalence. In *Foundations of Software Science and Computation Structures (FOSSACS'06)*. 398–412.
- ABADI, M. AND CORTIER, V. 2006. Deciding knowledge in security protocols under equational theories. *Theoretical Computer Science* 387, 1-2, 2–32.
- ABADI, M. AND FOURNET, C. 2001. Mobile values, new names, and secure communication. In *28th ACM Symposium on Principles of Programming Languages (POPL'01)*. ACM, 104–115.
- ANANTHARAMAN, S., NARENDHAN, P., AND RUSINOWITCH, M. 2007. Intruders with caps. In *18th International Conference on Term Rewriting and Applications (RTA'07)*. LNCS, vol. 4533. Springer.
- ARAPINIS, M., CHOTHIA, T., RITTER, E., AND RYAN, M. 2009. Untraceability in the applied pi calculus. In *Proceeding of the 1st International Workshop on RFID Security and Cryptography*.
- ARNAUD, M., CORTIER, V., AND DELAUNE, S. 2007. Combining algorithms for deciding knowledge in security protocols. In *Proc. 6th International Symposium on Frontiers of Combining Systems (FroCoS'07)*. Lecture Notes in Artificial Intelligence, vol. 4720. Springer, 103–117.
- BAUDET, M. 2005. Deciding security of protocols against off-line guessing attacks. In *12th ACM Conference on Computer and Communications Security (CCS'05)*. ACM Press, 16–25.
- BAUDET, M. 2007. Thèse de doctorat. Ph.D. thesis, Laboratoire Spécification et Vérification, ENS Cachan, France.
- BAUDET, M., CORTIER, V., AND DELAUNE, S. 2009. YAPA: A generic tool for computing intruder knowledge. In *20th International Conference on Rewriting Techniques and Applications (RTA'09)*. Lecture Notes in Computer Science, vol. 5595. Springer, Brasília, Brazil, 148–163.

- BAUDET, M., CORTIER, V., AND KREMER, S. 2005. Computationally sound implementations of equational theories against passive adversaries. In *32nd International Colloquium on Automata, Languages and Programming (ICALP'05)*. LNCS, vol. 3580. Springer, 652–663.
- BLANCHET, B. 2001. An Efficient Cryptographic Protocol Verifier Based on Prolog Rules. In *14th Computer Security Foundations Workshop (CSFW'01)*. IEEE Comp. Soc. Press, 82–96.
- BLANCHET, B., ABADI, M., AND FOURNET, C. 2008. Automated verification of selected equivalences for security protocols. *Journal of Logic and Algebraic Programming* 75, 1, 3–51.
- CHEVALIER, Y., KÜSTERS, R., RUSINOWITCH, M., AND TURUANI, M. 2003. An NP decision procedure for protocol insecurity with XOR. In *18th IEEE Symposium on Logic in Computer Science (LICS'03)*. IEEE Comp. Soc. Press.
- CIOBÂCĂ, Ș., DELAUNE, S., AND KREMER, S. 2009. Computing knowledge in security protocols under convergent equational theories. In *Proceedings of the 22nd International Conference on Automated Deduction (CADE'09)*, R. Schmidt, Ed. Lecture Notes in Artificial Intelligence. Springer, Montreal, Canada, 355–370.
- COMON-LUNDH, H. AND SHMATIKOV, V. 2003. Intruder deductions, constraint solving and insecurity decision in presence of exclusive or. In *18th IEEE Symposium on Logic in Computer Science (LICS'03)*. IEEE Comp. Soc. Press.
- CORIN, R., DOUMEN, J., AND ETALLE, S. 2004. Analysing password protocol security against off-line dictionary attacks. In *2nd International Workshop on Security Issues with Petri Nets and other Computational Models (WISP'04)*. ENTCS.
- CORTIER, V. AND DELAUNE, S. 2007. Deciding knowledge in security protocols for monoidal equational theories. In *14th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR'07)*. LNAI. Springer.
- CORTIER, V., DELAUNE, S., AND LAFOURCADE, P. 2006. A survey of algebraic properties used in cryptographic protocols. *Journal of Computer Security* 14, 1, 1–43.
- DELAUNE, S. AND JACQUEMARD, F. 2004. A decision procedure for the verification of security protocols with explicit destructors. In *11th ACM Conference on Computer and Communications Security (CCS'04)*. 278–287.
- DELAUNE, S., KREMER, S., AND RYAN, M. D. 2009. Verifying privacy-type properties of electronic voting protocols. *Journal of Computer Security* 17, 4 (July), 435–487.
- LOWE, G. 1996. Breaking and fixing the Needham-Schroeder public-key protocol using FDR. In *Tools and Algorithms for the Construction and Analysis of Systems (TACAS'96)*. LNCS, vol. 1055. Springer-Verlag, 147–166.
- MILLEN, J. AND SHMATIKOV, V. 2001. Constraint solving for bounded-process cryptographic protocol analysis. In *8th ACM Conference on Computer and Communications Security (CCS'01)*.

## A. APPENDIX

LEMMA 5.2 COMPLETENESS, SYNTACTIC DEDUCTION. *Let  $(\Phi, \Psi)$  be a state,  $M_0 \triangleright t_0 \in \Phi$ . Let  $N, t$  be two terms such that  $t \in \text{st}(t_0)$  and  $N \triangleright_\Phi t$ . Then there exists  $(\Phi', \Psi')$  and  $N'$  such that:*

- $(\Phi, \Psi) \Longrightarrow^* (\Phi', \Psi')$  using **B** rules, and
- $N' \triangleright t \in \Phi'$  and  $\Psi' \models N \bowtie N'$ .

PROOF. By hypothesis, we have that  $N \triangleright_\Phi t$ . This means that there exists a public context  $C$  and some facts  $M_1 \triangleright t_1, \dots, M_n \triangleright t_n \in \Phi$  such that  $N = C[M_1, \dots, M_n]$  and  $t = C[t_1, \dots, t_n]$ . Let  $C$  be such a context whose size is minimal. We show the result by structural induction on  $C$ .

*Base case:*  $C$  is reduced to an hole. Let  $(\Phi', \Psi') = (\Phi, \Psi)$  and  $N' = N$ . The result trivially holds.

*Induction step:*  $C = f(C_1, \dots, C_r)$  with  $f \in \mathcal{F}_{\text{pub}}$  of arity  $r$ . In such a case, we have  $t = f(u_1, \dots, u_r)$  and  $C_i[M_1, \dots, M_n] \triangleright_\Phi u_i$  with  $u_i \in \text{st}(t_0)$  for each  $1 \leq i \leq r$ . Thus, we can apply our induction hypothesis. We deduce that there exists  $(\Phi_1, \Psi_1)$  and terms  $N'_1, \dots, N'_r$  such that:

- $(\Phi, \Psi) \Longrightarrow^* (\Phi_1, \Psi_1)$  using **B** rules,
- $N'_i \triangleright u_i \in \Phi_1$  and  $\Psi_1 \models C_i[M_1, \dots, M_n] \bowtie N'_i$  for each  $1 \leq i \leq r$ .

From this we easily deduce that  $\Psi_1 \models N \bowtie f(N'_1, \dots, N'_r)$ . We apply one **B** rule. We have that  $M_0 \triangleright t_0, N'_1 \triangleright u_1, \dots, N'_r \triangleright u_r \in \Phi_1$ ,  $t = f(u_1, \dots, u_r) \in \text{st}(t_0)$  and  $f \in \mathcal{F}_{\text{pub}}$ . We distinguish two cases:

*Rule B.1:* Assume that for all  $M_t$  we have that  $(M_t \triangleright t) \notin \Phi_1$ .

Let  $\Phi' = \Phi_1 \cup \{f(N'_1, \dots, N'_r) \triangleright t\}$ ,  $\Psi' = \Psi_1$  and  $N' = f(N'_1, \dots, N'_r)$ . In order to conclude it remains to show that  $\Psi' \models N \bowtie N'$ . This is an easy consequence of the fact that  $\Psi_1 \models N \bowtie f(N'_1, \dots, N'_r)$ .

*Rule B.2.* Assume that there exists  $M_t$  such that  $M_t \triangleright t \in \Phi_1$ .

Let  $\Phi' = \Phi_1$ ,  $\Psi' = \Psi_1 \cup \{f(N'_1, \dots, N'_r) \bowtie M_t\}$  and  $N' = M_t$ . In order to conclude it remains to show that  $\Psi' \models N \bowtie N'$ . We have  $\Psi' \models f(N'_1, \dots, N'_r) \bowtie N'$  and  $\Psi' \models N \bowtie f(N'_1, \dots, N'_r)$ . This allows us to conclude.  $\square$

LEMMA 5.3 COMPLETENESS, SYNTACTIC EQUATIONS. *Let  $(\Phi, \Psi)$  be a state, and  $M, N$  be two terms such that  $M \triangleright_\Phi t$  and  $N \triangleright_\Phi t$  for some term  $t$ . Then there exists  $(\Phi', \Psi')$  such that:*

- $(\Phi, \Psi) \Longrightarrow^* (\Phi', \Psi')$  using **B** rules, and
- $\Psi' \models M \bowtie N$ .

PROOF. By hypothesis, we have that  $M \triangleright_\Phi t$  and  $N \triangleright_\Phi t$  for some term  $t$ . By definition of  $\triangleright_\Phi$ , we have that

- $M = C[M_1, \dots, M_k]$ ,  $N = C'[N_1, \dots, N_\ell]$  for some contexts  $C, C'$ ,
- the facts  $M_1 \triangleright t_1, \dots, M_k \triangleright t_k$  and  $N_1 \triangleright u_1, \dots, N_\ell \triangleright u_\ell$  are in  $\Phi$ ,
- $C[t_1, \dots, t_k] = C'[u_1, \dots, u_\ell]$ .

We prove the result by structural induction on  $C$  and  $C'$ . We assume w.l.o.g. that  $C$  is smaller than  $C'$  (in terms of number of symbols).

*Base case:*  $C$  is reduced to an hole. We have that  $C[M_1, \dots, M_k] = M_1$ . By hypothesis, we have that  $N \triangleright_{\Phi} t = t_1$  and thus  $t \in \text{st}(t_1)$ . Thanks to Lemma 5.2, there exists  $(\Phi', \Psi')$  and  $N'$  such that  $(\Phi, \Psi) \implies^* (\Phi', \Psi')$  using a **B** rule,  $N' \triangleright t_1 \in \Phi'$  and  $\Psi' \models N \bowtie N'$ . Since  $M_1 \triangleright t_1$  and  $N' \triangleright t_1$  are both in  $\Phi'$ , we deduce that  $N' = M_1$ . Hence we have that  $N' = M$  and thus we easily conclude.

*Induction step:*  $C = f(C_1, \dots, C_r)$  and  $C' = f(C'_1, \dots, C'_r)$  where  $f \in \mathcal{F}_{\text{pub}}$  is a symbol of arity  $r$  and  $C_1, \dots, C_r, C'_1, \dots, C'_r$  are contexts. Moreover, we have that  $C_i[t_1, \dots, t_k] = C'_i[u_1, \dots, u_\ell]$  for every  $1 \leq i \leq r$ . By applying the induction hypothesis, we deduce that there exists  $(\Phi', \Psi')$  such that

- $(\Phi, \Psi) \implies^* (\Phi', \Psi')$ , and
- $\Psi' \models C_i[M_1, \dots, M_k] \bowtie C'_i[N_1, \dots, N_\ell]$  for every  $1 \leq i \leq r$ .

Hence, we have that  $\Psi' \models M \bowtie N$ . This allows us to conclude.  $\square$

The following lemma justifies the notion of decomposition (Definition 4.1) as far as completeness is concerned.

LEMMA A.1 DECOMPOSITION OF A CONTEXT REDUCTION. *Let  $\Phi$  be a frame,  $l$  a (plain) term,  $\sigma$  a substitution, and  $M$  a term such that  $M \triangleright_{\Phi} l\sigma$ . Then there exist*

- a  $(n, p, q)$ -decomposition  $D$  of  $l$ , written  $l = D[l_1, \dots, l_n, y_1, \dots, y_{p+q}]$ ,
- $n$  deduction facts  $M_1 \triangleright t_1, \dots, M_n \triangleright t_n$  in  $\Phi$ ,
- $p + q$  recipes  $N_1, \dots, N_{p+q}$

such that

- for every  $1 \leq i \leq n$ ,  $t_i = l_i\sigma$  and
- for every  $1 \leq j \leq p + q$ ,  $N_j \triangleright_{\Phi} y_j\sigma$ .

In particular,  $D[M_1, \dots, M_n, N_1, \dots, N_{p+q}] \triangleright_{\Phi} l\sigma$ .

Besides, if  $l$  is a left-hand side of rule in  $\mathcal{R}$  and  $\Phi$  is  $\mathcal{R}$ -reduced,  $D$  is a proper decomposition (i.e.  $D \neq w_1$ ).

PROOF. Since  $M \triangleright_{\Phi} l\sigma$ , by definition there exists  $C$  and  $M_1^0 \triangleright t_1^0, \dots, M_m^0 \triangleright t_m^0$  in  $\Phi$  such that  $M = C[M_1^0, \dots, M_m^0]$  and  $l\sigma = C[t_1^0, \dots, t_m^0]$ .

Let  $x_1, \dots, x_m$  be fresh variables. Given that  $C[x_1, \dots, x_m]$  and  $l$  unify and have distinct variables, there exists a largest common context  $D_0$  such that  $l = D_0[l_1^0, \dots, l_a^0, y_1^0, \dots, y_b^0]$  and  $C = D_0[w_{j_1}, \dots, w_{j_a}, D_1, \dots, D_b]$  where the terms  $l_i^0$  are not variables and  $D_0$  uses all his parameters: in particular  $l\sigma = C[t_1^0, \dots, t_m^0]$  means that

- for every  $1 \leq k \leq a$ ,  $l_k^0\sigma = t_{j_k}^0$ , and
- for every  $1 \leq k \leq b$ ,  $y_k^0\sigma = D_k[t_1^0, \dots, t_m^0]$

Let  $n$  be the cardinal of  $\{l_1^0, \dots, l_a^0\}$ . For each distinct  $l_i$  in  $\{l_1^0, \dots, l_a^0\}$  ( $1 \leq i \leq n$ ), we choose  $k$  in  $\{1, \dots, a\}$  such that  $l_i = l_k^0$  and define  $M_i = M_k^0$  and  $t_i = l_k^0\sigma = l_i\sigma$ . Besides, for every  $k'$  such that  $l_{k'}^0 = l_k^0$ , we define  $w_{k'} = w_k$ .

Let  $p$  be the cardinal of  $\{y_1^0, \dots, y_b^0\} \cap \text{var}(l_1, \dots, l_n)$ . For each distinct  $y_j$  in  $\{y_1^0, \dots, y_b^0\}$  ( $1 \leq j \leq p$ ), we choose  $k$  in  $\{1, \dots, b\}$  such that  $y_j = y_k^0$  and define  $N_j = D_k[M_1^0, \dots, M_m^0]$ . Besides, for every  $k'$  such that  $y_{k'}^0 = y_k^0$ , we define  $w_{a+k'} = w_{p+j}$ .

Let  $q = b - p$ . We repeat the same operation for each distinct  $y_j$  in  $\{y_1^0, \dots, y_b^0\} - \text{var}(l_1, \dots, l_n)$  ( $p+1 \leq j \leq p+q$ ).

Finally, we let  $D = D_0[w_1, \dots, w_{a+b}]$ . By construction, we have that

- $l = D[l_1, \dots, l_n, y_1, \dots, y_{p+q}]$ ,
- the  $l_i$  are mutually distinct non-variable terms and the  $y_i$  are mutually distinct variables.
- $y_i \in \text{var}(l_1, \dots, l_n)$  iff  $i \leq p$ .
- $M_i \triangleright t_i$  is in  $\Phi$ ,
- for every  $1 \leq i \leq n$ ,  $t_i = l_i\sigma$ , and
- for every  $1 \leq j \leq p+q$ ,  $N_j \triangleright_{\Phi} y_j\sigma$ .

As for the last sentence, if  $D$  is a parameter, so is  $D_0$ . As  $l = y_k^0$  is impossible for a convergent system  $\mathcal{R}$ , we have  $D_0 = w_k$  with  $k \leq a$ . Hence  $C = w_{j_k}$  and  $t_k^0 = C[t_1^0, \dots, t_k^0] = l\sigma$  is not  $\mathcal{R}$ -reduced.  $\square$

**LEMMA 5.4 COMPLETENESS, CONTEXT REDUCTION.** *Let  $(\Phi, \Psi)$  be a state and  $M, t, t'$  be three terms such that  $M \triangleright_{\Phi} t$  and  $t \rightarrow_{\mathcal{R}} t'$ . Then, either  $(\Phi, \Psi) \Longrightarrow^* \perp$  or there exist  $(\Phi', \Psi')$ ,  $M'$  and  $t''$  such that*

- $(\Phi, \Psi) \Longrightarrow^* (\Phi', \Psi')$ ,
- $M' \triangleright_{\Phi'} t''$  with  $t' \rightarrow_{\mathcal{R}}^* t''$ , and
- $\Psi' \models M \bowtie M'$ .

*Besides, in both cases, the corresponding derivation from  $(\Phi, \Psi)$  can be chosen to consist of a number of **B** rules, possibly followed by one instance of **A** rule involving the same rewrite rule  $l \rightarrow r$  as the rewrite step  $t \rightarrow_{\mathcal{R}} t'$ .*

**PROOF.** By hypothesis, there exist a (public) context  $C$  and some deduction facts  $M_1^0 \triangleright t_1^0, \dots, M_{m_0}^0 \triangleright t_{m_0}^0 \in \Phi$  such that  $M = C[M_1^0, \dots, M_{m_0}^0]$  and  $t = C[t_1^0, \dots, t_{m_0}^0]$ .

Moreover, there exist a position  $\alpha$ , a substitution  $\sigma$  and a rewrite rule  $l \rightarrow r \in \mathcal{R}$  such that  $t|_{\alpha} = l\sigma$  and  $t' = t[r\sigma]_{\alpha}$ .

We note that  $\alpha$  must be a (symbol) position of  $C$  since the  $t_i^0$  are  $\mathcal{R}$ -reduced. Hence we may write  $C|_{\alpha}[t_1^0, \dots, t_{m_0}^0] = l\sigma$ .

By Lemma A.1, we deduce that there exist

- a proper  $(n, p, q)$ -decomposition  $D$  of  $l$ :  $l = D[l_1, \dots, l_n, y_1, \dots, y_p, z_1, \dots, z_q]$ ,
- $M_1 \triangleright t_1, \dots, M_n \triangleright t_n$  in  $\Phi$ ,
- $N_1, \dots, N_{p+q}$

such that

- for every  $1 \leq i \leq n$ ,  $t_i = l_i\sigma$ ,
- for every  $1 \leq j \leq p$ ,  $N_j \triangleright_{\Phi} y_j\sigma$ , and



—for every  $1 \leq k \leq q$ ,  $N_{p+k} \triangleright_{\Phi} z_k \sigma$ .

In particular, we obtain that

$$\begin{aligned} M|_{\alpha} &= C|_{\alpha}[M_1^0, \dots, M_{m_0}^0] \triangleright_{\Phi} C|_{\alpha}[t_1^0, \dots, t_{m_0}^0] = l\sigma \\ D[M_1, \dots, M_n, N_1, \dots, N_{p+q}] &\triangleright_{\Phi} D[t_1, \dots, t_n, y_1\sigma, \dots, y_p\sigma, z_1\sigma, \dots, z_q\sigma] = l\sigma \end{aligned}$$

Thus, by Lemma 5.3, there exists a derivation  $(\Phi, \Psi) \Longrightarrow^* (\Phi_1, \Psi_1)$  using **B** rules such that  $\Psi_1 \models M|_{\alpha} \bowtie D[M_1, \dots, M_n, N_1, \dots, N_{p+q}]$ .

Besides, since  $y_j$  belongs to  $\text{var}(l_1, \dots, l_n)$  by definition of decompositions,  $y_j\sigma$  is a subterm of some  $l_i\sigma = t_i$ . Since  $N_j \triangleright_{\Phi} y_j\sigma$ , by applying Lemma 5.2 repeatedly, we deduce that there exist some term  $M_{n+1}, \dots, M_{n+p}$  and a derivation  $(\Phi_1, \Psi_1) \Longrightarrow^* (\Phi_2, \Psi_2)$  using **B** rules such that for all  $j$ ,

— $M_{n+j} \triangleright y_j\sigma$  is in  $\Phi_2$ , and

— $\Psi_2 \models M_{n+j} \bowtie N_j$ .

Let  $N = D[M_1, \dots, M_{n+p}, N_{p+1}, \dots, N_{p+q}]$ . We deduce that  $N \triangleright_{\Phi_2} l\sigma$ , and

$$\Psi_2 \models M|_{\alpha} \bowtie D[M_1, \dots, M_n, N_1, \dots, N_{p+q}] \bowtie N$$

We now consider the application to  $(\Phi_2, \Psi_2)$  of a **A** rule that involves the rewrite rule  $l \rightarrow r$ , the decomposition  $D$ , the plain terms  $(t_1, \dots, t_{n+p}) = (l_1, \dots, l_n, y_1, \dots, y_p)\sigma$  and the substitution  $\sigma' = \sigma|_V$  obtained by restricted the  $\sigma$  to the domain  $V = \text{var}(l_1, \dots, l_n) = \text{var}(l_1, \dots, l_n, y_1, \dots, y_p)$ .

**Case A.3.** If  $(r\sigma') \downarrow_{\mathcal{R}}$  is not ground and  $\text{Ctx}(\Phi_2^+ \vdash_{\mathcal{R}}^? r\sigma') = \perp$  where  $\Phi_2^+ = \Phi_2 \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\}$ , then we may conclude that  $(\Phi_2, \Psi_2) \Longrightarrow \perp$  by an instance of rule **A.3** involving  $l \rightarrow r$ , the decomposition  $D$  and the facts  $M_1 \triangleright t_1, \dots, M_{n+p} \triangleright t_{n+p}$ .

**Case A.1.** If there exists  $N_0 = \text{Ctx}(\Phi_2^+ \vdash_{\mathcal{R}}^? r\sigma')$  where  $\Phi_2^+ = \Phi_2 \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\}$ . By Property (b) of  $\text{Ctx}$ , let  $s_0$  be such that  $N_0 \triangleright_{\Phi_2 \cup \{z_1, \dots, z_q\}} s_0$  and  $r\sigma' \rightarrow_{\mathcal{R}}^* s_0$ , and define

— $\Phi' = \Phi_2$ ,

— $\Psi' = \Psi_2 \cup \{\forall z_1, \dots, z_q. D[M_1, \dots, M_{n+p}, z_1, \dots, z_q] \bowtie N_0\}$ ,

— $M' = M[M_0]_{\alpha}$  where  $M_0 = N_0 \{z_i \mapsto N_{p+i}\}_{1 \leq i \leq q}$ ,

— $t'' = t[t_0]_{\alpha} = t'[t_0]_{\alpha}$  where  $t_0 = s_0 \{z_i \mapsto z_i\sigma\}_{1 \leq i \leq q}$ .

By construction, we have  $(\Phi_2, \Psi_2) \Longrightarrow (\Phi', \Psi')$  by an instance of rule **A.1**.

Besides,  $r\sigma' \rightarrow_{\mathcal{R}}^* s_0$  implies  $t'|_{\alpha} = r\sigma \rightarrow_{\mathcal{R}}^* t_0$  and  $t' \rightarrow_{\mathcal{R}}^* t''$ .

Given that  $\alpha \in \text{pos}(C)$  (where  $C$  is the previously context related to  $M \triangleright_{\Phi} t$ ) and  $M_0 \triangleright_{\Phi'} t_0$ , we have that  $M' = M[M_0]_{\alpha} \triangleright_{\Phi'} t[t_0]_{\alpha} = t''$ .

It remains to show that  $\Psi' \models M \bowtie M'$ . Indeed, we have seen that  $\Psi_2 \models M|_{\alpha} \bowtie N$  where  $N = D[M_1, \dots, M_{n+p}, z_1, \dots, z_q] \{z_i \mapsto N_{p+i}\}_{1 \leq i \leq q}$ . Besides, by definition of  $\Psi'$ , it holds that  $\Psi' \supseteq \Psi_2 \supseteq \Psi_1$  and we have that  $\Psi' \models D[M_1, \dots, M_{n+p}, z_1, \dots, z_q] \bowtie N_0$ . Therefore,  $\Psi' \models M|_{\alpha} \bowtie M_0$  and  $\Psi' \models M \bowtie M[M_0]_{\alpha} = M'$ .

**Case A.2:** if  $(r\sigma') \downarrow_{\mathcal{R}}$  is ground and  $\text{Ctx}(\Phi_2^+ \vdash_{\mathcal{R}}^? r\sigma') = \perp$  where  $\Phi_2^+ = \Phi_2 \cup \{z_1 \triangleright z_1, \dots, z_q \triangleright z_q\}$ , define

— $M_0 = D[M_1, \dots, M_{n+p}, \mathbf{a}, \dots, \mathbf{a}]$  and  $t_0 = (r\sigma') \downarrow_{\mathcal{R}}$ ,

- $\Phi' = \Phi_2 \cup \{M_0 \triangleright t_0\}$ ,
- $\Psi' = \Psi_2 \cup \{\forall z_1, \dots, z_q. D[M_1, \dots, M_{n+p}, z_1, \dots, z_q] \bowtie M_0\}$ ,
- $M' = M[M_0]_\alpha$ , and
- $t'' = t[t_0]_\alpha$ .

where  $\mathbf{a}$  is the fixed public constant of rule **A.2**.

By construction,  $(\Phi, \Psi) \implies (\Phi', \Psi')$  by an instance of the **A.2** rule.

Since  $t_0$  is ground and  $\sigma = \sigma'\sigma$ , we have  $t_0 = (r\sigma)\downarrow_{\mathcal{R}}$ . Therefore  $t' = t[r\sigma]_\alpha \rightarrow_{\mathcal{R}}^* t[(r\sigma)\downarrow_{\mathcal{R}}]_\alpha = t''$ .

Given that  $\alpha \in \text{pos}(C)$  and by construction  $M_0 \triangleright_{\Phi'} t_0$ , we have  $M' \triangleright_{\Phi'} t''$ .

It remains to show that  $\Psi' \models M \bowtie M'$ . Indeed, we have seen that  $\Psi_2 \models M|_\alpha \bowtie N$  where  $N = D[M_1, \dots, M_{n+p}, z_1, \dots, z_q]\{z_i \mapsto N_{p+i}\}_{1 \leq i \leq q}$ . By definition of  $\Psi'$ , it holds that  $\Psi' \models N \bowtie M_0$  hence  $\Psi' \models M \bowtie M[N]_\alpha \bowtie M[M_0]_\alpha = M'$ .

The additional properties claimed on the derivation are clear from the construction above.  $\square$